On the weak solutions to the Maxwell-Landau-Lifshitz equations and to the Hall-Magneto-Hydrodynamic equations

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Abstract

In this paper we deal with weak solutions to the Maxwell-Landau-Lifshitz equations and to the Hall-Magneto-Hydrodynamic equations. First we prove that these solutions satisfy some weak-strong uniqueness property. Then we investigate the validity of energy identities. In particular we give a sufficient condition on the regularity of weak solutions to rule out anomalous dissipation. In the case of the Hall-Magneto-Hydrodynamic equations we also give a sufficient condition to guarantee the magneto-helicity identity. Our conditions correspond to the same heuristic scaling as the one introduced by Onsager in hydrodynamic theory. Finally we examine the sign, locally, of the anomalous dissipations of weak solutions obtained by some natural approximation processes.

Keywords: Maxwell-Landau-Lifshitz equation, Hall-Magneto-Hydrodynamic equation, weak-strong uniqueness, dissipation, suitable weak solutions.

MSC: 35B99, 35Q60, 35Q35.

Contents

1 Introduction

In this paper we consider two non stationary quasilinear systems of PDEs originating from two different physical contexts, for which we develop a similar mathematical analysis. These systems are the Maxwell-Landau-Lifshitz equations and the Hall-Magneto-Hydrodynamic equations. When studying the associated Cauchy problem, weak solutions can be constructed, and they satisfy energy inequalities, which are equalities if the solutions are smooth.

First we prove that these solutions satisfy some weak-strong uniqueness property. More precisely we state that strong solutions are unique among the class of weak solutions. A key point here is precisely that the weak solutions considered satisfy some energy inequalities.

Then we investigate, according to the regularity of solutions, if the energy inequalities are equalities, or if some anomalous dissipation shows up during evolution. In order to prove that, under some regularity assumption on the solution, no dissipation occurs, we essentially analyze the commutation between regularization operators and nonlinearities. Of course energy identities rely on the particular structure of the nonlinearities of the systems under study, and a regularization of the equations may destroy this structure so that some cancellations arising in the formal operations leading to the energy identities are not true anymore. For both equations only quadratic nonlinearities are involved, when the equations are written in conservative form. In the case of the Hall-Magneto-Hydrodynamic equations we also consider some helicity identities.

Finally we examine, locally, the sign of anomalous dissipations. One motivation here is that one expects physical solutions effectively dissipate (and do not create) energy, globally as well as locally.

1.1 Presentation of the two systems

We start with a presentation of the two systems. In both cases we consider that the underlying physical space is the three dimensional euclidian space $\mathbb{R}^3$.

- **The Maxwell-Landau-Lifshitz equations (MLL for short),** which describes the coupling between the electromagnetic field and a magnetizable medium, see [7] and [41] for Physics references:

$$\partial_t m = m \times (\Delta m + H) - m \times \Big(m \times (\Delta m + H) \Big), \tag{1}$$
$$\partial_t H + \operatorname{curl} E = -\partial_t m, \tag{2}$$
$$\partial_t E - \operatorname{curl} H = 0, \tag{3}$$
$$\operatorname{div} E = \operatorname{div}(H + m) = 0. \tag{4}$$

Here $m(t,x)$ stands for the magnetic moment and takes values in the unit sphere S^2 of $\mathbb{R}^3$, whereas $H(t,x)$ and $E(t,x)$ are respectively the magnetic and electric fields.

- **The Hall-Magneto-Hydrodynamic equations (HMHD for short)** from Plasma Physics, see [42]:

$$\partial_t u + u \cdot \nabla u + \nabla p = (\operatorname{curl} B) \times B + \Delta u \tag{5}$$
$$\operatorname{div} u = 0, \tag{6}$$
$$\partial_t B - \operatorname{curl}(u \times B) + \operatorname{curl} \Big((\operatorname{curl} B) \times B \Big) = \Delta B, \tag{7}$$
$$\operatorname{div} B = 0, \tag{8}$$

where $u(t,x)$ and $B(t,x)$ are the fluid velocity and magnetic induction.

The paper [1] provides a derivation of this system from a two-fluid isothermal Euler-Maxwell system for electrons and ions.

The system (5)-(8) is a variant of the following **Magneto-Hydrodynamic system** with resistance and dissipation (**MHD for short**)

$$\partial_t u + u \cdot \nabla u + \nabla p = (\operatorname{curl} B) \times B + \Delta u, \tag{9}$$
$$\operatorname{div} u = 0, \tag{10}$$
$$\partial_t B - \operatorname{curl}(u \times B) = \Delta B, \tag{11}$$
$$\operatorname{div} B = 0. \tag{12}$$

Let us stress that the only difference is that the system (5)-(8) contains an extra-term in the left-hand-side of (7), $\operatorname{curl}\big((\operatorname{curl} B) \times B\big)$ which takes the Hall effect into account. This effect is believed to be the key for understanding the problem of magnetic reconnection which is involved in geomagnetic storms and solar flares, see for instance [39]. The Hall effect has also been studied in connection with the kinematic dynamo problem [34].

1.2 A few formal identities

In both cases existence of global weak solutions is known. We will recall precisely these results below, but let us emphasize here and now that their proof uses in a crucial way some energy bounds. Indeed a few formal computations lead to the following energy identities.

Energy identity for the MLL equations:

$$\forall T \geqslant 0, \quad \mathcal{E}_{\mathrm{MLL}}(T) + \int_0^T \mathcal{D}_{\mathrm{MLL}}(t)\, dt = \mathcal{E}_{\mathrm{MLL}}(0), \tag{13}$$

where

$$\mathcal{E}_{\mathrm{MLL}}(t) := \int_{\mathbb{R}^3} \Big(|E|^2(t,x) + |H|^2(t,x) + |\nabla m|^2(t,x) \Big) dx \quad \text{and} \quad \mathcal{D}_{\mathrm{MLL}}(t) := \int_{\mathbb{R}^3} |\partial_t m|^2(t,x)\, dx.$$

Energy identity for the MHD and HMHD equations:

$$\forall T \geqslant 0, \quad \mathcal{E}_{\mathrm{MHD}}(T) + \int_0^T \mathcal{D}_{\mathrm{MHD}}(t)\, dt = \mathcal{E}_{\mathrm{MHD}}(0) \quad \text{and} \quad \mathcal{E}_{\mathrm{HMHD}}(T) + \int_0^T \mathcal{D}_{\mathrm{HMHD}}(t)\, dt = \mathcal{E}_{\mathrm{HMHD}}(0), \tag{14}$$

where we denote

$$\mathcal{E}_{\mathrm{MHD}}(t) = \mathcal{E}_{\mathrm{HMHD}}(t) := \frac{1}{2} \int_{\mathbb{R}^3} \Big(|u|^2 + |B|^2 \Big)(t,x)\, dx,$$

$$\mathcal{D}_{\mathrm{MHD}}(t) = \mathcal{D}_{\mathrm{HMHD}}(t) := \int_{\mathbb{R}^3} \Big(|\operatorname{curl} u|^2 + |\operatorname{curl} B|^2 \Big)(t,x)\, dx.$$

These two formal identities can be justified when the solutions involved are smooth. However the weak solutions alluded here are obtained as weak limits of smooth functions so that only an inequality can be justified.

Another interesting quantity for the MHD and HMHD equations is the magneto-helicity

$$\mathcal{H}_m(t) := \int_{\mathbb{R}^3} \big(B \cdot A \big)(t,x)\, dx,$$

where A is a vector potential of B, that is a vector field satisfying $\operatorname{curl} A = B$. Indeed since B is divergence free the magneto-helicity is independent of the choice of the vector potential. In what follows we consider the gauge choice $\operatorname{div} A = 0$.

Magneto-helicity identity for the MHD and HMHD equations:

$$\forall T \geqslant 0, \quad \mathcal{H}_m(T) + \int_0^T \mathcal{D}_m(t)\, dt = \mathcal{H}_m(0), \tag{15}$$

where

$$\mathcal{D}_m(t) := 2 \int_{\mathbb{R}^3} \big(B \cdot (\operatorname{curl} B)\big)(t, x)\, dx.$$

Once again this formal identity can be justified for smooth solutions, but not for weak solutions, a priori.

One may also consider the fluid helicity

$$\mathcal{H}_f(t) := \int_{\mathbb{R}^3} \big(u \cdot \omega\big)(t, x)\, dx,$$

where $\omega := \operatorname{curl} u$ denotes the vorticity of the fluid. One then has formally the following identity.

Fluid helicity identity for the MHD and HMHD equations:

$$\forall T \geqslant 0, \quad \mathcal{H}_f(T) + \int_0^T \mathcal{D}_f(t)\, dt = \mathcal{H}_f(0), \tag{16}$$

where

$$\mathcal{D}_f(t) := 2 \int_{\mathbb{R}^3} \big(\omega \cdot \big((B \times \operatorname{curl} B) + \operatorname{curl} \omega\big)(t, x)\, dx.$$

In the case of the MHD system it is interesting to consider the crossed fluid-magneto-helicity:

$$\mathcal{H}_{fm}(t) := \int_{\mathbb{R}^3} \big(B \cdot u\big)(t, x)\, dx = \int_{\mathbb{R}^3} \big(A \cdot \omega\big)(t, x)\, dx,$$

which satisfies formally the following identity.

Crossed fluid-magneto-helicity identity for the MHD equations:

$$\forall T \geqslant 0, \quad \mathcal{H}_{fm}(T) + \int_0^T \mathcal{D}_{fm}(t)\, dt = \mathcal{H}_{fm}(0), \tag{17}$$

where

$$\mathcal{D}_{fm}(t) := 2 \int_{\mathbb{R}^3} \big(\omega \cdot \big((B \times \operatorname{curl} B) + \operatorname{curl} \omega\big)(t, x)\, dx.$$

Finally, in the case of the HMHD equations, a rather simple identity is obtained if one considers the total fluid-magneto-helicity:

$$\mathcal{H}_{f+m}(t) := \int_{\mathbb{R}^3} \big(u(t, x) + A(t, x)\big) \cdot \big(\omega(t, x) + B(t, x)\big)\, dx.$$

Total fluid-magneto-helicity identity for the HMHD equations:

$$\forall T \geqslant 0, \quad \mathcal{H}_{f+m}(T) + \int_0^T \mathcal{D}_{f+m}(t)\,dt = \mathcal{H}_{f+m}(0), \tag{18}$$

where

$$\mathcal{D}_{f+m}(t) := 2 \int_{\mathbb{R}^3} \left((\omega + B) \cdot \mathrm{curl}(\omega + B) \right)(t,x)\,dx.$$

1.3 A common structure

Let us emphasize that the identities (13)-(14)-(15)-(16)-(17)-(18) have the same form:

$$(\mathcal{E} \text{ or } \mathcal{H})(T) + \int_0^T \mathcal{D}(t)\,dt = (\mathcal{E} \text{ or } \mathcal{H})(0), \tag{19}$$

where the terms $\mathcal{E}$ or $\mathcal{H}$ denote respectively various energies and helicities, and the term $\mathcal{D}$ can be interpreted as some "dissipation", even if we do not claim anything about its sign in general at this point of the paper.

Actually, these global identities are obtained by space-time integration from local identities of the form:

$$\partial_t(e \text{ or } h) + d + \mathrm{div}\,f = 0, \tag{20}$$

where the terms e, h and d denote respectively various energy, helicity and dissipation densities, and f denotes some flux density. The global quantities $\mathcal{E}$, $\mathcal{H}$ and $\mathcal{D}$ are then obtained from e, h and d by integration in space, that is:

$$\mathcal{E}(t) := \int_{\mathbb{R}^3} e(t,x)\,dx, \quad \mathcal{H}(t) := \int_{\mathbb{R}^3} h(t,x)\,dx \text{ and } \mathcal{D}(t) := \int_{\mathbb{R}^3} d(t,x)\,dx. \tag{21}$$

The appropriate densities will be given explicitly in each case in Section 2.3.

1.4 Physical motivations

The investigation of the validity of energy or helicity identities for weak solutions to MLL and MHD equations is quite natural mathematically but is also linked to a few physical motivations that we want to address now.

The LLM equations. Physically, singularities of the magnetization field are referred to as Bloch points (see [50]). The mathematical analysis of these singularities has not been performed yet, but they may be of the same type as the ones of the heat flow. For the heat flow of maps from a manifold to the sphere S^2, at least when the space variable x belongs to some 2-dimensional manifold, (space-time) singularities of weak solutions correspond to the "bubbling" phenomenon, *i.e.* the asymptotic convergence of the solution, up to renormalization, towards some harmonic map (see [38] for a review on this topic): this "bubble" can be interpreted as the precise loss of energy of the solution at the singularity. The Landau-Lifshitz equation is close to this class of equations, in the following sense. Considering the simplified case, dropping the gyroscopic term $m \times \Delta m$ (and with no magnetic field), the equation

$$\partial_t m = -m \times (m \times \Delta m)$$

may be rewritten, for smooth solutions,

$$\partial_t m = \Delta m + |\nabla m|^2 m,$$

using $|m|^2 = 1$. This is the heat flow equation for maps with values in the sphere S^2, so that the Landau-Lifshitz equation (1) (with $H = 0$) can be viewed as a perturbation of this heat flow equation (on the other hand, the Landau-Lifshitz equation $\partial_t m = m \times \Delta m$, with no Gilbert dissipation term, is a Schrödinger map equation, as can be seen thanks to the stereographic projection – see [49]).

The HMHD equations. In ideal MHD, instead of considering equation (7) or equation (11), one considers the equation:

$$\partial_t B - \mathrm{curl}(u \times B) = 0. \tag{22}$$

If one introduces the flow η associated with the divergence free fluid velocity field u, that is the volume-preserving diffeomorphism $\eta(t)$ obtained by solving the ordinary differential equation: $\partial_t \eta = u(t, \eta)$ with initial data $\eta(0, x) = x$, equation (22) is (formally) equivalent to

$$B(t, \cdot) = \Big((D\eta)(t, \cdot)\, B(0, \cdot) \Big) (\eta(t, \cdot)^{-1}).$$

This means that any motion of the medium transports the magnetic field through a diffeomorphism action preserving the relative position of the field lines. The term "frozen-in" has been coined in this context. The topological structure of such a field, including its degree of knottedness, does not change along time evolution. In particular the helicity of a field, which measures the average linking of the field lines, or their relative winding, cf. [3], is preserved under the action of a volume-preserving diffeomorphism. A formal way to visualize this relies on Smirnov' decomposition of divergence free vector fields into elementary solenoids, cf. [48]. More precisely if we decompose initially the field B_0 into a superposition of elementary solenoids

$$\mathcal{S}_0 := \int_{\mathbb{T}} (\partial_s \tau_0(s)) \delta(x - \tau_0(s)) ds,$$

where $\mathbb{T}$ denotes the one-dimensional torus and $\mathbb{T} \ni s \mapsto \tau_0(s) \in \mathbb{R}^3$ denotes a loop, then, when time goes by, the field B is obtained as a superposition of the elementary solenoids

$$\mathcal{S} := \int_{\mathbb{T}} (\partial_s \tau(t, s)) \delta(x - \tau(t, s)) ds, \tag{23}$$

where $\mathbb{T} \ni s \mapsto \tau(t, s) \in \mathbb{R}^3$ denotes the loop obtained by solving $\partial_t \tau(t, s) = u(t, \tau(t, s))$, with the initial data $\tau(0, s) = \tau_0(s)$. Thus one sees that the corresponding loops cannot be unknotted without contradicting that the flow is a diffeomorphism.

Now if one takes into account the Hall effect, by considering the equation

$$\partial_t B - \mathrm{curl}(u \times B) + \mathrm{curl}\Big((\mathrm{curl}\, B) \times B \Big) = 0,$$

the previous analysis remains true if one substitutes to η the flow associated with the divergence free vector field $u - \mathrm{curl}\, B$.

However it appears that for a correct description of magnetic reconnection it is necessary to take into account the magnetic viscosity, as this is done here by considering equation (7). This implies that the field B is not simply transported (as a 2-form). In this reconnection process a subtle interplay takes place between the Hall effect and the magnetic viscosity [39].

Another motivation for the investigation of the topological structure of the magnetic field is that it provides obstructions to the full dissipation of the magnetic energy in stars or planets. In particular it has been shown by Arnold and Khesin [3] that helicity bounds from below the energy. The helicity approach to magnetic energy minoration in terms of the topology of magnetic lines has been generalized by Freedman and He [33] by introducing the notion of asymptotic crossing number.

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1.5 An analogy with Onsager's conjecture

The validity of conservation laws for weak solutions is a quite general issue in PDEs. In particular such a question was raised for incompressible flows by Onsager in [43]. The conjecture states that the minimal space regularity needed for a weak solution to the incompressible Euler equation to conserve energy is $1/3$, that is every weak solution to the Euler equations with Hölder continuous velocity of order $h > 1/3$ does not dissipate energy; and conversely, there exists a weak solution to the incompressible Euler equations of smoothness of exactly $1/3$ which does not conserve energy.

Concerning the first part of the conjecture, there was a renewal of interest starting with a paper by Eyink [30] who also discussed the connections of Onsager's conjecture with phenomenological approaches of fully-developed turbulence. Soon after this, Constantin, E and Titi gave a simple proof in [21] that a weak solution $u(t, x)$ of the incompressible Euler equations satisfying the condition

$$\lim_{y \to 0} \frac{1}{|y|} \int_0^T \int_{\mathbb{R}^3} |u(t, x) - u(t, x - y)|^3 \, dx \, dt = 0 \tag{24}$$

verifies the energy equality.

Let us stress that the condition (24) is a Besov type condition rather than a Hölder one. Actually the result in [21] is stated for a velocity in $L^3(0, T; B^\alpha_{3,\infty}(\Omega))$ with $\alpha > \frac{1}{3}$, but the proof works as well under the slightly weaker condition (24), see also [19, 29, 47].

Regarding the other part of the conjecture, the celebrated works by Scheffer [44] and Shnirelman [45] prove that there are nontrivial distributional solutions to the Euler equations which are compactly supported in space and time, and which therefore do not conserve the kinetic energy. Recently these results were extended by De Lellis and Szekelyhidi in [26] where they prove that there exist infinitely many compactly supported bounded weak solutions to the incompressible Euler equations. Consequently the existence of solutions, better than bounded, but with a regularity slightly weaker than (24), which dissipate the kinetic energy, was proved in a series of papers culminating in [8].

On the other hand in [5] Bardos and Titi prove that there exist some solutions to the incompressible Euler equations which do not satisfy (24) but which still preserve the energy. Indeed these solutions are some very explicit shear flows with only L^2 regularity.

The issue of the conservation of helicity for incompressible flows was tackled by [13, 19]. In particular Theorem 4.2 in [19] proves the validity of helicity conservation for solutions to the incompressible Euler equation which are in $L^\infty(0, T; H^{\frac{1}{2}}(\mathbb{R}^3))$ and satisfy

$$\lim_{y \to 0} \frac{1}{|y|^2} \int_0^T \int_{\mathbb{R}^3} |u(t, x) - u(t, x - y)|^3 \, dx \, dt = 0. \tag{25}$$

We wish to mention that some anisotropic versions of the Onsager conjecture were studied by Caflisch, Klapper and Steele in [16] and by Shvydkoy in [46]. Furthermore, the papers [20, 32] deal with the first part of the Onsager conjecture for the incompressible Navier-Stokes equations when the fluid occupies a domain limited by a boundary.

Finally, a related phenomenon is described in the recent papers [22, 23] by Dascaliuc and Grujić, who study the energy cascade in the physical space for both the Euler and Navier-Stokes equations.

1.6 Structure of the paper

In the next section we start with a reminder of the weak theories available for the MLL and HMHD equations and we present our results. Then we state some weak-strong uniqueness results for these solutions. Next we list the local counterparts of the formal identities given in the introduction, and we establish some local conservation identities corresponding to (13), (14) and (15). These identities include in general, i.e. for weak solutions, some anomalous dissipation terms. Then we provide a regularity condition, of Besov type, which is sufficient for the vanishing of these anomalous dissipation terms. Finally we investigate the sign of these energy anomalous dissipations. In Sections 3 and 4, we prove the weak-strong uniqueness results for the MLL and HMHD equations, respectively. In Section 5 we provide the proof of the first part of these results (local conservations) for both the MLL and HMHD equations. The proof of the other part (vanishing of anomalous dissipations) requires a few more technicalities which are given in Section 6. Then we provide in Section 7 a regularity condition sufficient for the vanishing of anomalous energy dissipation for the MLL equation. In Section 8, we show the vanishing of magneto-helicity and energy anomalous dissipations for the HMHD equations under analogous conditions. Section 9 is devoted to the crossed fluid-magneto-helicity identity for the MHD equations. In Section 10 we prove the results stated in Section 2 about the sign of the energy dissipation for weak solutions to the MLL and HMHD equations obtained by standard processes. An Appendix is devoted to the proof of a technical Bernstein-type lemma for a time-space Besov space involved in the analysis.

2 Presentation of the results

2.1 A reminder of the weak theories for the MLL and HMHD equations

Existence of global weak solutions to the MLL equations. Let us first recall that the MLL system admits some global weak solutions. This result relies on the following conservative form of (1), sometimes referred to as the Gilbert form of the equations:

$$\partial_t m + m \times \partial_t m = 2 \sum_i \partial_i \left(m \times \partial_i m \right) + 2m \times H, \tag{26}$$

where the sum is over $1, 2, 3$.

For smooth functions, the two equations, (1) and (26), are equivalent, but the last form is more appropriate for some u with weak regularity. Indeed we have the following result of existence of weak solutions to the MLL equations, see [51, 10, 37] (see also, concerning weak solutions for the Landau-Lifshitz equation, the papers [2, 36]); see also the book [35], as well as references therein.

Theorem 1. *Let be given m_0 in $L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ such that $|m_0| = 1$ almost everywhere and ∇m_0 is in $L^2(\mathbb{R}^3; \mathbb{R}^9)$, E_0 and H_0 in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ such that $\operatorname{div} E_0 = \operatorname{div}(H_0 + m_0) = 0$. Then, there exists (m, E, H) : $(0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^9$ such that, for all $T > 0$,*

$$m \in L^\infty((0, T) \times \mathbb{R}^3; \mathbb{R}^3), \quad |m| = 1 \ a.e.,$$
$$\nabla m \in L^\infty((0, T); L^2(\mathbb{R}^3; \mathbb{R}^9)) \quad and \quad \partial_t m \in L^2((0, T) \times \mathbb{R}^3; \mathbb{R}^3),$$
$$(E, H) \in L^\infty((0, T); L^2(\mathbb{R}^3; \mathbb{R}^6)),$$

and (m, E, H) is a weak solution to the MLL equations (26)-(2)-(3)-(4) on $(0, \infty) \times \mathbb{R}^3$, with initial value (m_0, E_0, H_0). Moreover, this solution satisfies the following energy inequality,

$$for \ almost \ every \ T > 0, \quad \mathcal{E}_{\mathrm{MLL}}(T) + \int_0^T \mathcal{D}_{\mathrm{MLL}}(t) \, dt \leqslant \mathcal{E}_{\mathrm{MLL}}(0). \tag{27}$$

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Existence of global weak solutions to the HMHD equations. Let us now tackle the case of the HMHD equations. These equations are recast in a conservative form:

$$\partial_t u + \text{div}(u \otimes u - B \otimes B) + \nabla p_m = \Delta u, \tag{28}$$
$$\text{div}\, u = 0, \tag{29}$$
$$\partial_t B - \text{curl}(u \times B) + \text{curl}\,\text{div}(B \otimes B) = \Delta B, \tag{30}$$
$$\text{div}\, B = 0, \tag{31}$$

where p_m denotes the magnetic pressure

$$p_m := p + \frac{1}{2}|B|^2.$$

Let

$$\mathcal{H} := \{\phi \in L^2(\mathbb{R}^3; \mathbb{R}^3) \mid \text{div}\,\phi = 0\} \quad \text{and} \quad \mathcal{V} := \{\phi \in H^1(\mathbb{R}^3; \mathbb{R}^3) \mid \text{div}\,\phi = 0\}.$$

In the recent paper [14], Chae, Degond and Liu establish the existence of global weak solutions for the incompressible viscous resistive HMHD model written as follows:

Theorem 2 (Chae-Degond-Liu, Acheritogaray-Degond-Frouvelle-Liu)**.** *Let u_0 and B_0 be in $\mathcal{H}$. Then there exists a global weak solution (u, B) to the HMHD model (28)-(31), corresponding to these initial data. Moreover, for all $T > 0$, we have*

$$(u, B) \in \left(L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V})\right)^2, \tag{32}$$

and this solution satisfies the following energy inequality:

$$\text{for almost every } T > 0, \quad \mathcal{E}_{\text{HMHD}}(T) + \int_0^T \mathcal{D}_{\text{HMHD}}(t)\, dt \leqslant \mathcal{E}_{\text{HMHD}}(0). \tag{33}$$

Actually the first result concerning existence of weak solutions to the HMHD system is due to Acheritogaray, Degond, Frouvelle and Liu [1], who prove Theorem 2 in a periodic setting.

2.2 Weak-Strong uniqueness

One major issue about the weak solutions mentioned above is their uniqueness. In particular regarding the Landau-Lifshitz equations, non-uniqueness of weak solutions is proved in [2]. On the other hand, up to our knowledge, uniqueness of weak solutions to the HMHD equations has not been proved or disproved yet.

Still, one way to get uniqueness results is to consider stronger solutions. Actually, for both the MLL and HMHD equations, there also exists some results about the local-in-time existence and uniqueness of strong solutions. Let us mention here the papers [11, 12] for the MLL equations and [14] for the HMHD equations.

Facing these two theories, the weak one and the strong one, it is natural to wonder if there is a weak-strong uniqueness principle. Indeed, such a property ensures that the weak theory is an extension of the strong one, rather than a bifurcation.

The following results provide such properties for both the MLL and HMHD equations. In both cases a key point is that weak solutions satisfy an energy inequality. This echoes the similar well-known results for the incompressible Navier-Stokes and Euler equations, cf. for example, respectively, [17] and [27, Proposition 1]. Let us also mention here, in this direction, the recent extension to the full Navier-Stokes-Fourier system by [31].

Let us warn the reader that we will not try here to minimize the smoothness of the strong solutions involved in the following statement.

Weak-Strong uniqueness for the MLL equations. Our first result states that a strong solution is unique among the class of weak solutions, as given by Theorem 1.

Theorem 3. *Consider initial data* (m_0, E_0, H_0) *as in Theorem 1, and assume moreover that they are smooth.*
Finally assume that

- $(m_2, E_2, H_2) : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^9$ *is a global weak solution to the MLL equations (26)-(2)-(3)-(4) on* $(0, T) \times \mathbb{R}^3$*, with initial value* (m_0, E_0, H_0)*, as given by Theorem 1;*

- $(m_1, E_1, H_1) : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^9$ *is a smooth solution to the MLL equations (26)-(2)-(3)-(4), up to some time* $T > 0$*, also with the initial value* (m_0, E_0, H_0)*.*

Then $(m_2, E_2, H_2) = (m_1, E_1, H_1)$ *on* $(0, T) \times \mathbb{R}^3$*.*

Let us mention that, up to our knowledge, this result is already new for the Landau-Lifshitz equations:

$$\partial_t m + m \times \partial_t m = 2m \times \Delta m. \tag{34}$$

Actually we will first give a proof of the corresponding result for the global weak solution to the Landau-Lifshitz equations, as given in [2, Theorem 1.4], and then we will give the proof of Theorem 3.

Let us therefore state here the case of the Landau-Lifshitz equation (34).

Theorem 4. *Consider an initial data* m_0 *in* $L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ *such that* $|m_0| = 1$ *almost everywhere and such that* ∇m_0 *is in* $L^2(\mathbb{R}^3; \mathbb{R}^9)$*. Assume moreover that* m_0 *is smooth. Finally assume that*

- m_2 *is a global weak solution of (34) on* $(0, \infty) \times \mathbb{R}^3$ *satisfying the energy inequality: for almost every* $T \geqslant 0$*,*

$$J_{\mathrm{LL}}[m_2](T) := \big(\int_{\mathbb{R}^3} |\nabla m_2|^2 \, dx \big)(T) + \int_0^T \int_{\mathbb{R}^3} |\partial_t m_2|^2 \, dx \, dt \leqslant \int_{\mathbb{R}^3} |\nabla m_0|^2 \, dx. \tag{35}$$

- m_1 *is a smooth solution to the Landau-Lifshitz equation (34) up to some time* $T > 0$*, with the same initial data* m_0*.*

Then $m_2 = m_1$ *on* $(0, T) \times \mathbb{R}^3$*.*

Weak-Strong uniqueness for the HMHD equations. Let us now turn to the case of the HMHD equations.

Theorem 5. *Consider initial data* u_0 *and* B_0 *in* $\mathcal{H}$*, and assume moreover that they are smooth.*
Finally assume that

- (u_2, B_2) *is a global weak solution to the HMHD model (28)-(31), associated with the initial data* (u_0, B_0)*, as in Theorem 3;*

- (u_1, B_1) *is a smooth solution the HMHD model (28)-(31) on* $(0, T)$*, for some* $T > 0$*, also associated with the initial data* (u_0, B_0)*.*

Then $(u_2, B_2) = (u_1, B_1)$ *on* $(0, T) \times \mathbb{R}^3$*.*

We will prove Theorem 5 in a simplified setting which focuses on the difficulty due to the Hall effect. Actually the corresponding statement for the MHD equations is well-known, see for instance [24], and the extension to the general case by combining the corresponding proof and the proof below for the simplified Hall model is routine.

2.3 Local energy and helicity identities

In this section we recall the explicit formulations of the formal local identities hinted in the introduction. We start with the energy identities.

Local energy identity for the MLL equations: formally,

$$\partial_t e_{\mathrm{MLL}} + d_{\mathrm{MLL}} + \operatorname{div} f_{\mathrm{MLL}} = 0, \tag{36}$$

where

$$e_{\mathrm{MLL}} := |E|^2 + |H|^2 + |\nabla m|^2, \quad d_{\mathrm{MLL}} := |\partial_t m|^2, \quad f_{\mathrm{MLL}} := -2(\partial_t m \cdot \partial_i m)_{i=1,2,3} + 2H \times E. \tag{37}$$

Local energy identity for the HMHD equations: formally,

$$\partial_t e_{\mathrm{HMHD}} + d_{\mathrm{HMHD}} + \operatorname{div} f_{\mathrm{HMHD}} = 0, \tag{38}$$

where

$$e_{\mathrm{HMHD}} := \frac{1}{2}\Big(|u|^2 + |B|^2\Big), \quad d_{\mathrm{HMHD}} := |\operatorname{curl} u|^2 + |\operatorname{curl} B|^2, \tag{39}$$

$$f_{\mathrm{HMHD}} := (\tfrac{1}{2}|u|^2 + p)u + B \times (u \times B) + (\operatorname{curl} u) \times u + (\operatorname{curl} B) \times B + \big((\operatorname{curl} B) \times B\big) \times B. \tag{40}$$

Let us mention here that, despite the global energy identities are the same for the HMHD and MHD equations, their local counterparts are different. Indeed, for the MHD equations, one has to drop out the last term in the flux density above.

Local energy identity for the MHD equations: formally,

$$\partial_t e_{\mathrm{MHD}} + d_{\mathrm{MHD}} + \operatorname{div} f_{\mathrm{MHD}} = 0, \tag{41}$$

where

$$e_{\mathrm{MHD}} := \frac{1}{2}\Big(|u|^2 + |B|^2\Big) = e_{\mathrm{HMHD}}, \quad d_{\mathrm{MHD}} := |\operatorname{curl} u|^2 + |\operatorname{curl} B|^2 = d_{\mathrm{HMHD}}, \tag{42}$$

but

$$f_{\mathrm{MHD}} := (\tfrac{1}{2}|u|^2 + p)u + B \times (u \times B) + (\operatorname{curl} u) \times u + (\operatorname{curl} B) \times B\Big). \tag{43}$$

Let us now tackle the helicity identities.

Local magneto-helicity identity for the HMHD equations: formally,

$$\partial_t h_{m,\mathrm{HMHD}} + d_{m,\mathrm{HMHD}} + \operatorname{div} f_{m,\mathrm{HMHD}} = 0, \tag{44}$$

where

$$h_{m,\mathrm{HMHD}} := A \cdot B, \; d_{m,\mathrm{HMHD}} := 2B \cdot \operatorname{curl} B, \; f_{m,\mathrm{HMHD}} := (2(u - \operatorname{curl} B) \times B - 2\operatorname{curl} B - \partial_t A) \times A. \tag{45}$$

Once again, despite the global magneto-helicity identities are the same for the HMHD and MHD equations, their local counterparts are not the same. Indeed, for the MHD equations, on has to drop out the last term in the flux density above.

Local magneto-helicity identity for the MHD equations: formally,

$$\partial_t h_{m,\mathrm{MHD}} + d_{m,\mathrm{MHD}} + \operatorname{div} f_{m,\mathrm{MHD}} = 0, \tag{46}$$

where

$$h_{m,\mathrm{MHD}} := A \cdot B = h_{m,\mathrm{HMHD}}, \quad d_{m,\mathrm{MHD}} := 2B \cdot \operatorname{curl} B = d_{m,\mathrm{HMHD}}, \tag{47}$$

but

$$f_{m,\mathrm{MHD}} := (2u \times B - 2\operatorname{curl} B - \partial_t A) \times A. \tag{48}$$

Local fluid helicity identity for the MHD and HMHD equations: formally,

$$\partial_t h_f + d_f + \operatorname{div} f_f = 0, \tag{49}$$

where

$$h_f := u \cdot \omega, \; d_f := 2\omega \cdot (\operatorname{curl}\omega + B \times \operatorname{curl} B), \; f_f := (\omega \cdot u)u + (p + \tfrac{1}{2}u^2)\omega - u \times (\operatorname{curl}\omega + B \times \operatorname{curl} B).$$

Local crossed fluid-magneto-helicity identity for the MHD equations: formally,

$$\partial_t h_{fm} + d_{fm} + \operatorname{div} f_{fm} = 0, \tag{50}$$

where

$$h_{fm} := u \cdot B, \; d_{fm} := 2\omega \cdot \operatorname{curl} B, \; f_{fm} := (u \cdot B)u + (p - \tfrac{1}{2}|u|^2)B + (\operatorname{curl} u) \times B + (\operatorname{curl} B) \times u. \tag{51}$$

Local total fluid-magneto-helicity identity for HMHD equations: formally,

$$\partial_t h_{f+m} + d_{f+m} + \operatorname{div} f_{f+m} = 0, \tag{52}$$

where

$$h_{f+m} := (u + A) \cdot (\omega + B), \; d_{f+m} := 2(\omega + B) \cdot \operatorname{curl}(\omega + B),$$
$$f_{f+m} := \Big(\partial_t(u + A) - 2u \times (\omega + B) + 2\operatorname{curl}(\omega + B)\Big) \times (u + A).$$

2.4 Regularization of quadratic terms

The above weak solutions are of course solutions on $(0, \infty) \times \mathbb{R}^3$ in the distributional sense. In the sequel, for a given such solution, we shall compare the difference between the equation, with each term regularized, and the same (linear or quadratic) terms obtained from the regularization of the solution. We thus introduce some notations.

Let $\psi \in C_c^\infty(\mathbb{R}^3; \mathbb{R})$ be nonnegative, and such that $\int_{\mathbb{R}^3} \psi(x) \, dx = 1$. For all $\varepsilon \in (0, 1)$, we define the usual mollifier $\psi^\varepsilon := \varepsilon^{-3}\psi(\cdot/\varepsilon)$. Then, for any function u on $\mathbb{R}^3$, we set

$$u_\varepsilon(x) = (\psi^\varepsilon * u)(x) = \int_{\mathbb{R}^3} \psi^\varepsilon(y)u(x - y)dy. \tag{53}$$

For all $\varepsilon \in (0, 1)$ and functions ϕ^1, ϕ^2, we also define

$$\mathcal{A}^\varepsilon[\phi^1, \phi^2] := (\phi^1 \cdot \phi^2)_\varepsilon - \phi_\varepsilon^1 \cdot \phi_\varepsilon^2, \tag{54}$$
$$\mathcal{B}^\varepsilon[\phi^1, \phi^2] := (\phi^1 \times \phi^2)_\varepsilon - \phi_\varepsilon^1 \times \phi_\varepsilon^2, \tag{55}$$
$$\mathcal{C}^\varepsilon[\phi^1, \phi^2] := (\phi^1 \otimes \phi^2)_\varepsilon - \phi_\varepsilon^1 \otimes \phi_\varepsilon^2. \tag{56}$$

2.5 Some Besov type conditions

Our goal is to provide some sufficient conditions, similar to (24), which rule out anomalous dissipation in the MLL and in the HMHD equations.

The Fourier transform $\mathcal{F}$ is defined on the space of integrable functions $f \in L^1(\mathbb{R}^3)$ by $(\mathcal{F}f)(\xi) := \int_{\mathbb{R}^3} e^{-2i\pi x \cdot \xi} f(x) dx$, and extended to an automorphism of the space $\mathcal{S}'(\mathbb{R}^3)$ of tempered distributions, which is the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ of rapidly decreasing functions. We consider the following extensions of condition (24).

Definition 6. *Let $T > 0$, $\alpha \in (0,1)$ and $p, r \in [1, \infty]$. We denote by $\mathcal{S}'_h$ the space of tempered distributions u on $(0,T) \times \mathbb{R}^3$ such that for all $\theta \in C_c^\infty(\mathbb{R}^3)$, there holds*

$$\|\theta(\lambda D)u\|_{L^\infty((0,T) \times \mathbb{R}^3)} \underset{\lambda \to \infty}{\longrightarrow} 0,$$

where $\theta(D)$ is the Fourier multiplier defined by $\theta(D)u = \mathcal{F}^{-1}(\theta \mathcal{F}u)$. For every function u on $(0,T) \times \mathbb{R}^3$ we define, for $(t, y) \in (0,T) \times (\mathbb{R}^3 \setminus \{0\})$,

$$f_{\alpha,p}[u](t,y) := \frac{\|u(t, \cdot - y) - u(t, \cdot)\|_{L^p(\mathbb{R}^3)}}{|y|^\alpha}.$$

We denote

- *by $\widetilde{L}^r(0,T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))$, the space of functions u on $(0,T) \times \mathbb{R}^3$, belonging to $\mathcal{S}'_h$, which satisfy*

$$\sup_y \|f_{\alpha,p}[u](\cdot, y)\|_{L^r(0,T)} < \infty,$$

  equipped with the seminorm

$$\|u\|_{\widetilde{L}^r(0,T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))} := \sup_y \|f_{\alpha,p}[u](\cdot, y)\|_{L^r(0,T)};$$

- *by $\widetilde{L}^r(0,T; \dot{B}^{\alpha+1}_{p,\infty}(\mathbb{R}^3))$, the subspace of the functions u in $\widetilde{L}^r(0,T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))$ which satisfy, for $i = 1, 2, 3$, $\partial_i u \in \widetilde{L}^r(0,T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))$, equipped with the seminorm*

$$\|u\|_{\widetilde{L}^r(0,T; \dot{B}^{\alpha+1}_{p,\infty}(\mathbb{R}^3))} := \|u\|_{\widetilde{L}^r(0,T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))} + \sum_{i=1}^3 \|\partial_i u\|_{\widetilde{L}^r(0,T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))};$$

- *by $\widetilde{L}^r(0,T; \dot{B}^\alpha_{p,c_0}(\mathbb{R}^3))$, the subspace of the functions u in $\widetilde{L}^r(0,T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))$ which satisfy*

$$\|f_{\alpha,p}[u](\cdot, y)\|_{L^r(0,T)} \to 0 \text{ when } y \to 0;$$

- *by $\widetilde{L}^r(0,T; \dot{B}^{\alpha+1}_{p,c_0}(\mathbb{R}^3))$, the subspace of the functions u in $\widetilde{L}^r(0,T; \dot{B}^\alpha_{p,c_0}(\mathbb{R}^3))$ which satisfy, for $i = 1, 2, 3$, $\partial_i u \in \widetilde{L}^r(0,T; \dot{B}^\alpha_{p,c_0}(\mathbb{R}^3))$;*

- *by $L^r(0,T; L^p(\mathbb{R}^3))_{\mathrm{loc}}$ the space of functions u on $(0,T) \times \mathbb{R}^3$ such that for all $\chi \in C_c^\infty((0,T) \times \mathbb{R}^3)$, χu belongs to $L^r(0,T; L^p(\mathbb{R}^3))$.*

- *by $\widetilde{L}^r(0,T; \dot{B}^\alpha_{p,\alpha}(\mathbb{R}^3))_{\mathrm{loc}}$, where α holds for ∞ or c_0, the space of functions u on $(0,T) \times \mathbb{R}^3$ such that for all $\chi \in C_c^\infty((0,T) \times \mathbb{R}^3)$, χu belongs to $\widetilde{L}^r(0,T; \dot{B}^\alpha_{p,\alpha}(\mathbb{R}^3))$.*

In particular, condition (24) is equivalent to $u \in \widetilde{L}^3(0,T; \dot{B}_{3,c_0}^{1/3}(\mathbb{R}^3))$.

The notation $\widetilde{L}$, rather than L, is used to emphasize the fact that time integration is performed before taking the supremum in y. This contrasts with the more classical space $L^r(0,T; \dot{B}_{p,\infty}^\alpha(\mathbb{R}^3))$, the space of the functions u on $(0,T) \times \mathbb{R}^3$ which satisfy $\|\sup_y f_{\alpha,p}[u](\cdot, y)\|_{L^r(0,T)} < \infty$, equipped with the semi-norm $\|u\|_{L^r(0,T; \dot{B}_{p,\infty}^\alpha(\mathbb{R}^3))} := \|\sup_y f_{\alpha,p}[u](\cdot, y)\|_{L^r(0,T)}$. It is not difficult to see that $L^r(0,T; \dot{B}_{p,\infty}^\alpha(\mathbb{R}^3)) \subset \widetilde{L}^r(0,T; \dot{B}_{p,\infty}^\alpha(\mathbb{R}^3))$. This kind of spaces has been introduced by Chemin and Lerner in [18].

2.6 Anomalous dissipation for the MLL equations

We have the following result.

Theorem 7. *Let (m, E, H) be a weak solution to the MLL equations (26)-(2)-(3)-(4) given by Theorem 1.*

Let $d_{\mathrm{MLL}}^{\mathfrak{a}}$ denotes the local anomalous energy dissipation for the MLL equations:

$$d_{\mathrm{MLL}}^{\mathfrak{a}} := \partial_t e_{\mathrm{MLL}} + d_{\mathrm{MLL}} + \operatorname{div} f_{\mathrm{MLL}}, \tag{57}$$

where $(e_{\mathrm{MLL}}, d_{\mathrm{MLL}}, f_{\mathrm{MLL}})$ is given by (37).

i) Then the local anomalous energy dissipation $d_{\mathrm{MLL}}^{\mathfrak{a}}$ can be obtained as follows. Let

$$d_{\mathrm{MLL}}^{\mathfrak{a},\varepsilon} := -\mathcal{B}^\varepsilon[m, \partial_t m - 2(H + \Delta m)] \cdot (\partial_t m_\varepsilon - 2(H_\varepsilon + \Delta m_\varepsilon)). \tag{58}$$

Then,

$$d_{\mathrm{MLL}}^{\mathfrak{a},\varepsilon} \to d_{\mathrm{MLL}}^{\mathfrak{a}} \text{ in } \mathcal{D}'\big((0,\infty) \times \mathbb{R}^3; \mathbb{R}\big) \text{ when } \varepsilon \to 0,$$

and this holds true whatever is the mollifier chosen in Section 2.4.

ii) Assume furthermore that m belongs to $\widetilde{L}^3(0,T; \dot{B}_{p,c_0}^\alpha(\mathbb{R}^3))_{\mathrm{loc}}$ for some $\alpha \in (3/2, 2)$ and

$$p := \frac{9}{3\alpha - 1}. \tag{59}$$

Then the local anomalous energy dissipation $d_{\mathrm{MLL}}^{\mathfrak{a}}$ vanishes.

The first part of the theorem provides a way (actually many, since the choice of the mollifier is arbitrary) to obtain the anomalous dissipation. The second part provides a sufficient condition on the regularity of the weak solution to guarantee that this anomalous dissipation vanishes. In this case the local energy identity (36) holds true and the global identity (13) as well.

We briefly describe the strategy of the proof of the above theorem. Let us consider here, to simplify, the case of the Landau-Lifshitz equation (34). Then the energy identity is formally obtained as follows. One takes the inner product of (34) with $\partial_t m$ and Δm to get

$$(\partial_t m)^2 = 2(m \times \Delta m) \cdot \partial_t m, \tag{60}$$

$$\partial_t m \cdot \Delta m + (m \times \partial_t m) \cdot \Delta m = 0. \tag{61}$$

Observe that the combination $(60) - 2(61)$ yields

$$(\partial_t m)^2 - 2\partial_t m \cdot \Delta m = 0.$$

Then integrate by parts in space and finally integrate in time to obtain the energy identity: for any $T > 0$,

$$\int_{\mathbb{R}^3} |\nabla m|^2(T, x) dx + \int_{(0,T) \times \mathbb{R}^3} |\partial_t m|^2 \, dx \, dt = \int_{\mathbb{R}^3} |\nabla m|^2(0, x) dx.$$

However some terms involved in this process do not even have a sense for weak solutions. In fact, we shall apply a smoothing convolution to equation (34), using the notation ε in the index as in (53), and then we shall take the inner product with the approximation $\partial_t m_\varepsilon$ and Δm_ε. Still some cancellations, which were trivial in the formal calculations above, are not guaranteed anymore, and the point is to be able to get rid of the spurious terms. For example when we take the inner product of the regularized version of (34) with Δm_ε we face in particular the expression

$$\int_{(0,T) \times \mathbb{R}^3} (m \times \Delta m)_\varepsilon \cdot \Delta m_\varepsilon \, dx \, dt. \tag{62}$$

Actually it appears in the proof below that this term is somehow the worse we have to cope with. The idea behind Theorem 7 is that the regularity assumption on m allows to get rid of the term (62) when ε goes to 0.

A formal argument consists in simply counting that in (62), there appear four derivatives, in a product of three terms. One can then think that the regularity threshold above which integration by parts becomes possible is $4/3$. For the Euler equations (where the quantity obtained in energy estimates is $((u \cdot \nabla)u) \cdot u)$, Onsager's conjecture precisely indicated the formal threshold $1/3$, which can be interpreted as the result of one derivative in a product of three terms. Here, for the MLL system, we are in some sense less able to "share out" the derivatives, and conclude only for a regularity above $3/2$.

However the couple of exponents (α, p) in Theorem 7 is critical, in the sense given by Shvydkoy in [47], referring to the following dimensional argument. Let M, X, T be respectively some units for magnetic moment, length and time. Then the quantity in (62) has a dimension equal to $X^{-1} T M^3$. On the other hand the quantity $\|f_{\alpha,p}[m](\cdot, y)\|_{L^r(0,T)}$ from Definition 6 has a dimension equal to

$$X^{-\alpha} \left(T(M^p X^3)^{\frac{r}{p}} \right)^{\frac{1}{r}} = X^{\frac{3}{p} - \alpha} T^{\frac{1}{r}} M.$$

We would like to control the term (62) by $\|f_{\alpha,p}[m](\cdot, y)\|^3_{L^r(0,T)}$ which has a dimension equal to $X^{\frac{9}{p} - 3\alpha} T^{\frac{3}{r}} M^3$, which provides $r = 3$ and (59).

In [5] the authors prove that there exist some solutions to the incompressible Euler equations which do not satisfy (24) but which still preserve the energy. It is quite easy to provide a similar result for the MLL equation. Indeed, omitting the magnetic field, it is sufficient to consider the example which is used in [2] in order to exhibit a case where weak solutions are non unique.

2.7 Anomalous dissipation for the HMHD equations

Let us denote by $K[\cdot]$ the Biot-Savart law in $\mathbb{R}^3$. We consider B given by Theorem 2 and $A := K[B]$, so that

$$A \in L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; H^2(\mathbb{R}^3)), \quad \text{and} \quad \text{curl } A = B.$$

Observe in particular that we consider here the gauge choice div $A = 0$.

Our second main result concerns the magneto-helicity conservation or dissipation for the HMHD equations.

Theorem 8. *Let (u, B) be a solution to the HMHD equations given by Theorem 2. Let us denote by $d_m^{\mathfrak{a}}$ the local magneto-helicity anomalous dissipation:*

$$d_m^{\mathfrak{a}} := \partial_t h_m + d_m + \operatorname{div} f_m, \tag{63}$$

where (h_m, d_m, f_m) is given by (45).

    i) Define

$$d_m^{\mathfrak{a},\varepsilon} := 2A_\varepsilon \cdot \operatorname{curl} \mathcal{B}^\varepsilon[u, B] - 2A_\varepsilon \cdot \operatorname{curl} \operatorname{div} \mathcal{C}^\varepsilon[B, B]. \tag{64}$$

    Then

$$d_m^{\mathfrak{a},\varepsilon} \to d_m^{\mathfrak{a}} \ in \ \mathcal{D}'\big((0, \infty) \times \mathbb{R}^3; \mathbb{R}\big) \ when \ \varepsilon \to 0,$$

    and this holds true whatever is the mollifier chosen in Section 2.4.

    ii) Assume furthermore that $B \in \widetilde{L}^3(0, T; \dot{B}_{3,c_0}^{\frac{1}{3}}(\mathbb{R}^3))_{\mathrm{loc}}$. Then the anomalous magneto-helicity dissipation $d_m^{\mathfrak{a}}$ vanishes.

Remark 1. *The proof of part ii) below will show that the first term in the definition (64) of $d_m^{\mathfrak{a},\varepsilon}$ converges to 0 when $\varepsilon \to 0^+$ in the sense of distributions under the sole assumption that (u, B) is given by Theorem 2.*

Investigation of the validity of the fluid helicity identity (16) and of the total fluid-magneto-helicity (18) is very similar and is left aside in this paper. Regarding energy, we have the following result.

Theorem 9. *Let (u, B) be a solution to the HMHD equations given by Theorem 2, and assume that $B \in L^4((0, T) \times \mathbb{R}^3)_{\mathrm{loc}}$. Let us denote by $d_{\mathrm{HMHD}}^{\mathfrak{a}}$ the local energy anomalous dissipation:*

$$d_{\mathrm{HMHD}}^{\mathfrak{a}} := \partial_t e_{\mathrm{HMHD}} + d_{\mathrm{HMHD}} + \operatorname{div} f_{\mathrm{HMHD}}, \tag{65}$$

where $(e_{\mathrm{HMHD}}, d_{\mathrm{HMHD}}, f_{\mathrm{HMHD}})$ is given by (39).

    i) Let

$$
\begin{aligned}
d_{\mathrm{HMHD}}^{\mathfrak{a},\varepsilon} \ :=\ & -u_\varepsilon \cdot \operatorname{div}\left(\mathcal{C}^\varepsilon[u, u] - \mathcal{C}^\varepsilon[B, B]\right) - \frac{1}{2} u_\varepsilon \cdot \nabla \mathcal{A}^\varepsilon[B, B] \\
& -B_\varepsilon \cdot \left(\operatorname{curl} \mathcal{B}^\varepsilon[u, B]\right) + B_\varepsilon \cdot \left(\operatorname{curl} \operatorname{div} \mathcal{C}^\varepsilon[B, B]\right).
\end{aligned}
$$

    Then

$$d_{\mathrm{HMHD}}^{\mathfrak{a},\varepsilon} \to d_{\mathrm{HMHD}}^{\mathfrak{a}} \ in \ \mathcal{D}'\big((0, \infty) \times \mathbb{R}^3; \mathbb{R}\big) \ when \ \varepsilon \to 0,$$

    and this holds true whatever is the mollifier chosen in Section 2.4.

ii) Assume furthermore that

$$u \in \widetilde{L}^3(0, T; \dot{B}_{3,c_0}^{\frac{1}{3}}(\mathbb{R}^3))_{\mathrm{loc}} \ and \ B \in \widetilde{L}^3(0, T; \dot{B}_{3,c_0}^{\frac{2}{3}}(\mathbb{R}^3))_{\mathrm{loc}}.$$

    Then the anomalous energy dissipation $d_{\mathrm{HMHD}}^{\mathfrak{a}}$ vanishes.

2.8 A comparison with the MHD equations

It is interesting to compare the results for the HMHD equations with the case of the Magneto-Hydrodynamic system without Hall effect. Recasting (9) under the conservative form

$$\partial_t u + \mathrm{div}(u \otimes u - B \otimes B) + \nabla p_m = \Delta u, \tag{66}$$

we can define the notion of weak solution. Actually this was observed a long time ago by Duvaut and Lions who proved the following theorem (see [24]).

Theorem 10 (Duvaut-Lions). *Let u_0 and B_0 be in $\mathcal{H}$ and $T > 0$. Then there exists a weak solution*

$$(u, B) \in \left(L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \right)^2,$$

for the MHD model (66)-(10)-(11)-(12) corresponding to these initial data. Moreover, this solution satisfies (33).

Now following the proofs of Theorem 8 and Theorem 9 we also have the following result about the conservation of energy, of magneto-helicity and of crossed fluid-magneto-helicity.

Theorem 11. *Let (u, B) be a solution to (66) given by Theorem 10.*

 i) *Then the local magneto-helicity identity (46) is valid.*

 ii) *Assume furthermore that*

- *u is in $\widetilde{L}^3(0, T; \dot{B}^\alpha_{3,\infty}(\mathbb{R}^3))_{\mathrm{loc}}$ with $\alpha \in (0, 1)$ or u is in $L^3((0, T) \times \mathbb{R}^3)_{\mathrm{loc}}$, and then we set $\alpha = 0$,*
- *B is in $\widetilde{L}^3(0, T; \dot{B}^\beta_{3,\infty}(\mathbb{R}^3))_{\mathrm{loc}}$ with $\beta \in (0, 1)$,*

 and at least one the three following properties holds true:

- *$\alpha = 0$,*
- *$\alpha \in (0, 1)$ and u is in $\widetilde{L}^3(0, T; \dot{B}^\alpha_{3,c_0}(\mathbb{R}^3))_{\mathrm{loc}}$,*
- *B is in $\widetilde{L}^3(0, T; \dot{B}^\beta_{3,c_0}(\mathbb{R}^3))_{\mathrm{loc}}$.*

 Finally assume that $\alpha + 2\beta \geqslant 1$. Then the local energy estimate (41) holds true.

 iii) *Let us consider again (u, B) a solution to (66) given by Theorem 10. Assume furthermore that*

- *u is in $\widetilde{L}^3(0, T; \dot{B}^\alpha_{3,\infty}(\mathbb{R}^3))_{\mathrm{loc}}$ with $\alpha \in (0, 1)$,*
- *B is in $\widetilde{L}^3(0, T; \dot{B}^\beta_{3,\infty}(\mathbb{R}^3))_{\mathrm{loc}}$ with $\beta \in (0, 1)$,*

 and at least one the two following properties holds true:

- *u is in $\widetilde{L}^3(0, T; \dot{B}^\alpha_{3,c_0}(\mathbb{R}^3))_{\mathrm{loc}}$,*
- *B is in $\widetilde{L}^3(0, T; \dot{B}^\beta_{3,c_0}(\mathbb{R}^3))_{\mathrm{loc}}$.*

 Finally assume that $2\alpha + \beta \geqslant 1$ and $3\beta \geqslant 1$. Then the local crossed fluid-magneto-helicity identity (50) holds true.

Therefore the Hall effect does not modify formally the laws of conservation of energy and magneto-helicity but it could be, in view of Theorem 8, Theorem 9 and Theorem 11, that it creates some extra anomalous dissipation in these laws for solutions which have only a quite bad regularity.

Observe that Theorem 11 extends to the case of the viscous resistive MHD some earlier results by [16] about the ideal MHD.

Remark 2. *We do not reproduce here the dimensional argument (given after Theorem 7) for the HMHD and MHD equations but one can check that the conditions given in Theorem 8, Theorem 9 and Theorem 11 are critical in the sense of this dimensional analysis.*

The proof of Part i) and Part ii) of Theorem 11 is left to the reader since it can be proved along the same lines as the proofs of Theorem 8 and Theorem 9. The proof of the Part iii) is tackled in Section 9.

2.9 Suitable weak solutions

Observe that despite we use the word "dissipation" in the statements above we do not claim anything about the sign of $e_{\mathrm{MLL}}^{\mathfrak{a}}$ or $e_{\mathrm{HMHD}}^{\mathfrak{a}}$. This terminology would be particularly appropriate if the distributions $e_{\mathrm{MLL}}^{\mathfrak{a}}$ or $e_{\mathrm{HMHD}}^{\mathfrak{a}}$ were non positive.

The corresponding feature for the Navier-Stokes equations has been quite useful in order to obtain partial regularity theorems limiting the parabolic Hausdorff dimension of the singular set, see [9]. In particular, in this context, the term "suitable" has been coined for weak solutions that have a non positive anomalous dissipation. Strikingly enough the approximation process used by Leray in order to establish the existence of weak solutions actually leads to suitable weak solutions, see [29]. In Leray's scheme, the approximate equations read:

$$\partial_t u^\varepsilon + (u^\varepsilon)_\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = \Delta u^\varepsilon, \tag{67}$$
$$\mathrm{div}\, u^\varepsilon = 0. \tag{68}$$

One may also argue that an appropriate sign condition on the anomalous dissipation could be helpful to select among weak solutions, which ones may be considered physically acceptable, as one might think that the lack of smoothness could lead to local energy creation. Indeed in the case of the inviscid Burgers equation in one space dimension the requirement to be suitable coincides with the usual entropy condition of negative jumps, which does imply uniqueness. However such a result has not been proved yet for the Navier-Stokes equations, up to our knowledge, and the case of the HMHD equations could be even more difficult. In the case of the MLL equations, such a result is even more desirable since Alouges and Soyeur have proved in [2] non-uniqueness of weak solutions to the Landau-Lifshitz equation.

In this section, we investigate what can be said about the sign of the anomalous energy dissipations $d_{\mathrm{MLL}}^{\mathfrak{a}}$ and $d_{\mathrm{HMHD}}^{\mathfrak{a}}$ for some rather standard processes used in order to prove the existence of weak solutions to the MLL equations and of the HMHD equations.

Case of the MLL equations. One difficulty in establishing the existence of weak solutions to the MLL equations as claimed in Theorem 1 is due to the condition $|m| = 1$ almost everywhere. A brutal application of the usual strategy of mollification of the equation fails to capture this constraint. A by-now usual way to overcome this difficulty consists in using a Ginzburg-Landau type penalization, following the analysis performed in [2] for the Landau-Lifshitz equation and [10] for the MLL equations. Here we will consider,

for $\varepsilon \in (0, 1)$, the penalized equations:

$$\partial_t m^\varepsilon - m^\varepsilon \times \partial_t m^\varepsilon = 2\Big(\Delta m^\varepsilon + H^\varepsilon - (H^\varepsilon \cdot m^\varepsilon)m^\varepsilon - \frac{1}{\varepsilon}(|m^\varepsilon|^2 - 1)m^\varepsilon\Big), \tag{69}$$

$$\partial_t H^\varepsilon + \operatorname{curl} E^\varepsilon = -\partial_t m^\varepsilon, \tag{70}$$

$$\partial_t E^\varepsilon - \operatorname{curl} H^\varepsilon = 0, \tag{71}$$

$$\operatorname{div} E^\varepsilon = \operatorname{div}(H^\varepsilon + m^\varepsilon) = 0. \tag{72}$$

Let us emphasize in particular that equation (69) is slightly different from the penalized equation used in [10]. The difference is that we add the term $(H^\varepsilon \cdot m^\varepsilon)m^\varepsilon$ in order to be able to apply the weak maximum principle and to get a better regularity. A similar idea was used in [28] and in [25] for the quasi-stationary Landau-Lifshitz equations.

As a first step in order to prove Theorem 1, one then establishes the existence of weak solutions to (69)-(72), an easy task since the condition $|m| = 1$ a.e. has been dropped out. More precisely, one obtains that for initial data as in Theorem 1, that is for m_0 in $L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ such that $|m_0| = 1$ almost everywhere and ∇m_0 is in $L^2(\mathbb{R}^3; \mathbb{R}^9)$, E_0 and H_0 in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ such that $\operatorname{div} E_0 = \operatorname{div}(H_0 + m_0) = 0$, there exists, for all $\varepsilon \in (0, 1)$, a weak solution $(m^\varepsilon, E^\varepsilon, H^\varepsilon) : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^9$ of (69)-(72) on $(0, \infty) \times \mathbb{R}^3$, with initial value (m_0, E_0, H_0), such that, for all $T > 0$,

$$m^\varepsilon \in L^\infty((0, T); \dot{H}^1(\mathbb{R}^3; \mathbb{R}^3)), \quad (E^\varepsilon, H^\varepsilon) \in L^\infty((0, T); L^2(\mathbb{R}^3; \mathbb{R}^6)),$$
$$(|m^\varepsilon|^2 - 1, \nabla m^\varepsilon) \in L^\infty((0, T); L^2(\mathbb{R}^3; \mathbb{R}^{10})) \quad \text{and} \quad \partial_t m^\varepsilon \in L^2((0, T) \times \mathbb{R}^3; \mathbb{R}^3),$$

and

$$\forall T \geqslant 0, \quad \Big(\mathcal{E}^\varepsilon_{\mathrm{MLL}}(T) + \mathcal{E}^\varepsilon_{\mathrm{GL}}(T) + \int_0^T \mathcal{D}^\varepsilon_{\mathrm{MLL}}(t)\, dt\Big)_\varepsilon \text{ is bounded uniformly,} \tag{73}$$

where

$$\mathcal{E}^\varepsilon_{\mathrm{MLL}}(t) := \int_{\mathbb{R}^3} \Big(|E^\varepsilon|^2(t, x) + |H^\varepsilon|^2(t, x) + |\nabla m^\varepsilon|^2(t, x)\Big) dx, \quad \mathcal{D}^\varepsilon_{\mathrm{MLL}}(t) := \int_{\mathbb{R}^3} |\partial_t m^\varepsilon|^2(t, x)\, dx,$$

$$\text{and} \quad \mathcal{E}^\varepsilon_{\mathrm{GL}}(t) := \frac{1}{2\varepsilon} \int_{\mathbb{R}^3} \Big(|m^\varepsilon(t, x)|^2 - 1\Big)^2 dx.$$

Then, by standard compactness arguments, one infers that there exists

$$m \in L^\infty((0, T); \dot{H}^1(\mathbb{R}^3; \mathbb{R}^3)), \quad (E, H) \in L^\infty((0, T); L^2(\mathbb{R}^3; \mathbb{R}^6)),$$
$$\text{with } (|m|^2 - 1, \nabla m) \in L^\infty((0, T); L^2(\mathbb{R}^3; \mathbb{R}^{10})) \quad \text{and} \quad \partial_t m \in L^2((0, T) \times \mathbb{R}^3; \mathbb{R}^3),$$

such that, up to a subsequence, m^ε weakly converges to m in $H^1((0, T) \times \mathbb{R}^3; \mathbb{R}^3)$, $|m^\varepsilon|^2 - 1$ converges to 0 weakly in $L^2((0, T) \times \mathbb{R}^3; \mathbb{R})$ and almost everywhere, $(E^\varepsilon, H^\varepsilon)$ converges to (E, H) weakly in $L^2((0, T) \times \mathbb{R}^3; \mathbb{R}^6)$. Moreover (m, E, H) is a weak solution as claimed in Theorem 1. Since we do not claim here any originality let us simply refer for instance to [2] where this step is detailed for the Landau-Lifshitz equation, and to [10] where the case of the MLL equations is tackled.

Our point here is the following.

Theorem 12. *Let (m, E, H) be a weak solution to the MLL equations obtained as a limit point of the sequence $(m^\varepsilon, E^\varepsilon, H^\varepsilon)$ as considered above. Assume moreover that, up to a subsequence, $H^\varepsilon \times E^\varepsilon$ and, for $i = 1, 2, 3$, $\partial_t m^\varepsilon \cdot \partial_i m^\varepsilon$ converge respectively to $H \times E$ and $\partial_t m \cdot \partial_i m$ in the sense of distributions, and that $H^\varepsilon \cdot m^\varepsilon$ converges in $L^2_{\mathrm{loc}}((0, T) \times (\mathbb{R}^3))$ to $H \cdot m$. Then there exist two non negative distributions $d^{\mathfrak{a},1}_{\mathrm{MLL}}$ and $e^{\mathfrak{a}}_{\mathrm{MLL}}$ such that $d^{\mathfrak{a}}_{\mathrm{MLL}} = -d^{\mathfrak{a},1}_{\mathrm{MLL}} - \partial_t e^{\mathfrak{a}}_{\mathrm{MLL}}$.*

This theorem can be seen somehow as a counterpart of [29, Proposition 4] which shows that any weak solution to the Euler equation which is a strong limit of suitable solutions to the Navier-Stokes equation, as viscosity goes to zero, is a suitable weak solution. Let us stress that in Theorem 12 the assumption is weaker than the strong convergence of H^ε, E^ε, $H^\varepsilon \cdot m^\varepsilon$ and ∇m^ε in $L^2_{\mathrm{loc}}((0,T) \times (\mathbb{R}^3))$. On the other hand when this strong convergence holds, the proof will reveal that $d^{\mathfrak{a}}_{\mathrm{MLL}}$ vanishes and that $e^{\mathfrak{a}}_{\mathrm{MLL}}$ is only due to the possible lack of strong convergence of the energy density

$$e^\varepsilon_{\mathrm{GL}} := \frac{1}{2\varepsilon}\big(|m^\varepsilon|^2 - 1\big)^2 \tag{74}$$

associated with $\mathcal{E}^\varepsilon_{\mathrm{GL}}$. It would be interesting to investigate the existence of another way to construct weak solutions to the MLL equations for which the distribution $e^{\mathfrak{a}}_{\mathrm{MLL}}$ vanishes as well.

Case of the HMHD equations. Mimicking the mollification process in (67)-(68), we consider the equations:

$$\partial_t u^\varepsilon + (u^\varepsilon)_\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = (\mathrm{curl}\, B^\varepsilon) \times (B^\varepsilon)_\varepsilon + \Delta u^\varepsilon \tag{75}$$
$$\mathrm{div}\, u^\varepsilon = 0, \tag{76}$$
$$\partial_t B^\varepsilon - \mathrm{curl}(u^\varepsilon \times (B^\varepsilon)_\varepsilon) + \mathrm{curl}\Big((\mathrm{curl}\, B^\varepsilon) \times (B^\varepsilon)_\varepsilon\Big) = \Delta B^\varepsilon, \tag{77}$$
$$\mathrm{div}\, B^\varepsilon = 0. \tag{78}$$

Standard arguments yield that for all u_0 and B_0 in $\mathcal{H}$, for all $\varepsilon \in (0,1)$, there exists a global weak solution

$$(u^\varepsilon, B^\varepsilon) \in \Big(L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V})\Big)^2$$

solution to (75)-(78) corresponding to these initial data. Moreover this solution satisfies the energy inequality (33). Therefore, up to a subsequence, u^ε and B^ε converge in $L^\infty(0,T;\mathcal{H})$ weak-* and weakly in $L^2(0,T;\mathcal{V})$ respectively to u and B. Using some a priori temporal estimates and Aubin-Lions' lemma, we deduce that, up to a subsequence, u^ε and B^ε converges in $L^3((0,T) \times \mathbb{R}^3)_{\mathrm{loc}}$. This allows to establish that (u,B) is a weak solution to the HMHD equations associated with the initial data (u_0, B_0). We refer to [1, 14] for the details of this procedure, though for a slightly different regularization scheme.

Our point here is the following.

Theorem 13. *Let (u,B) be a weak solution to the HMHD equations obtained as a limit point of the sequence $(u^\varepsilon, B^\varepsilon)$ as considered above. Assume moreover that, up to a subsequence, B^ε converges to B in $L^4((0,T) \times \mathbb{R}^3)_{\mathrm{loc}}$. Then the anomalous energy dissipation $d^{\mathfrak{a}}_{\mathrm{HMHD}}$ is non positive.*

Let us mention that the assumption that, up to a subsequence, B^ε converges to B in $L^4((0,T) \times \mathbb{R}^3)_{\mathrm{loc}}$, can be dropped out in the case of the MHD equations, as will be shown in the proof.

2.10 A few extra comments

An analysis relying on Littlewood-Paley decomposition as in [19] for the incompressible Euler equations can be transposed to the MLL and HMHD systems. We will not go in this direction here. Let us only mention that it seems that the same range of regularity is attained by both methods.

In [17], Chemin recently used a strategy which is precisely close to the Littlewood-Paley counterpart of the key lemma used in [21], and adapted below, cf. Section 6.2, in order to obtain some sharp weak-strong uniqueness result for the incompressible Navier-Stokes equations, see in particular Lemma 2.3 in [17].

It could therefore be interesting to see if it is possible to sharpen the weak-strong uniqueness results given in Theorem 3 and in Theorem 5, that is to extend the statements to rougher strong solutions, using for example the regularization approach.

Since the results obtained here are local in space, they could easily be adapted to the case where the equations are set in a bounded domain; one then obtains a sufficient condition for the dissipation to vanish in the interior of the domain. However the counterpart of such a result up to the boundary seems more difficult. In this direction let us mention the results [20, 32] about the incompressible Navier-Stokes equations. Yet, with respect to this issue the MLL and HMHD equations seem to be closer to the incompressible Euler equations, for which there is, to our knowledge, no result of regularity up to the boundary.

3 Weak-strong uniqueness: Proof of Theorem 4 and of Theorem 3

3.1 Case of the Landau-Lifshitz equation

For sake of expository, we shall first give the proof of Theorem 4 which deals with the case of the Landau-Lifshitz equation (34), where the fields E and H are omitted. The extension to the MLL equations (26)-(2)-(3)-(4) is given in the next subsection.

We therefore consider $m_2 : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ a global weak solution to (34) on $(0, \infty) \times \mathbb{R}^3$ satisfying the energy inequality (35), for almost every $T \geqslant 0$. Here the weak formulation of the Landau-Lifshitz equation (34) reads: for every $\Psi \in H^1((0, \infty) \times \mathbb{R}^3; \mathbb{R}^3)$,

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t m_2 + m_2 \times \partial_t m_2) \cdot \Psi \, dx \, dt = -2 \sum_i \int_0^T \int_{\mathbb{R}^3} (m_2 \times \partial_i m_2) \cdot \partial_i \Psi \, dx \, dt, \tag{79}$$

where the sum is over $1, 2, 3$. The initial data m_0 is prescribed in the trace sense, and is here assumed to be smooth.

Let us consider m_1, a smooth solution to (34) with the same initial data m_0. We denote $m := m_1 - m_2$ and expand $J_{\mathrm{LL}}[m](T)$ into

$$J_{\mathrm{LL}}[m](T) = J_{\mathrm{LL}}[m_1](T) + J_{\mathrm{LL}}[m_2](T) - 2\Big(\int_{\mathbb{R}^3} \nabla m_1 : \nabla m_2 \, dx\Big)(T) - 2\int_0^T \int_{\mathbb{R}^3} \partial_t m_1 \cdot \partial_t m_2 \, dx \, dt.$$

Using some integration by parts, we have

$$\Big(\int_{\mathbb{R}^3} \nabla m_1 : \nabla m_2 \, dx\Big)(T) \;=\; \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i \partial_t m_1) \cdot \partial_i m_2 \, dx \, dt \tag{80}$$

$$- \int_0^T \int_{\mathbb{R}^3} (\Delta m_1) \cdot \partial_t m_2 \, dx \, dt + \int_{\mathbb{R}^3} |\nabla m_0|^2 \, dx.$$

Now, the two solutions satisfy the energy inequality (35), so that, for almost every $T \geqslant 0$,

$$J_{\mathrm{LL}}[m](T) \leqslant K_{\mathrm{LL}}[m_1, m_2](T), \tag{81}$$

where

$$K_{\mathrm{LL}}[m_1, m_2](T) \;:=\; -2 \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i \partial_t m_1) \cdot \partial_i m_2 \, dx \, dt + 2 \int_0^T \int_{\mathbb{R}^3} (\Delta m_1) \cdot \partial_t m_2 \, dx \, dt$$

$$- 2 \int_0^T \int_{\mathbb{R}^3} \partial_t m_1 \cdot \partial_t m_2 \, dx \, dt. \tag{82}$$

We now use the weak formulation (79) with the test functions $\Psi = \partial_t m_1$ and $\Psi = -2\Delta m_1$:

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t m_2 + m_2 \times \partial_t m_2) \cdot \partial_t m_1 \, dx \, dt = -2 \sum_i \int_0^T \int_{\mathbb{R}^3} (m_2 \times \partial_i m_2) \cdot \partial_i \partial_t m_1 \, dx \, dt, \tag{83}$$

$$-2 \int_0^T \int_{\mathbb{R}^3} (\partial_t m_2 + m_2 \times \partial_t m_2) \cdot \Delta m_1 \, dx \, dt = 4 \sum_i \int_0^T \int_{\mathbb{R}^3} (m_2 \times \partial_i m_2) \cdot \partial_i \Delta m_1 \, dx \, dt. \tag{84}$$

On the other hand, since m_1 is a strong solution to (34), we have

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t m_1 + m_1 \times \partial_t m_1) \cdot \partial_t m_2 \, dx \, dt = 2 \sum_i \int_0^T \int_{\mathbb{R}^3} (m_1 \times \Delta m_1) \cdot \partial_t m_2 \, dx \, dt, \tag{85}$$

$$2 \int_0^T \int_{\mathbb{R}^3} \partial_i (\partial_t m_1 + m_1 \times \partial_t m_1) \cdot \partial_i m_2 \, dx \, dt = 4 \sum_i \int_0^T \int_{\mathbb{R}^3} \partial_i (m_1 \times \partial_i m_1) \cdot \partial_i m_2 \, dx \, dt. \tag{86}$$

Thanks to (83)-(86) we get that

$$K_{\mathrm{LL}}[m_1, m_2] := -4(I_1 + I_2) + 2(I_3 + \ldots + I_6) + (I_7 + I_8),$$

with

$$I_1 := \sum_i \int_0^T \int_{\mathbb{R}^3} (m_2 \times \partial_i m_2) \cdot \partial_i \Delta m_1 \, dx \, dt, \quad I_2 := \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i (m_1 \times \Delta m_1)) \cdot \partial_i m_2 \, dx \, dt,$$

$$I_3 := \sum_i \int_0^T \int_{\mathbb{R}^3} \partial_i (m_1 \times \partial_t m_1) \cdot \partial_i m_2 \, dx \, dt, \quad I_4 := -\int_0^T \int_{\mathbb{R}^3} (m_2 \times \partial_t m_2) \cdot \Delta m_1 \, dx \, dt,$$

$$I_5 := -\int_0^T \int_{\mathbb{R}^3} (m_1 \times \Delta m_1) \cdot \partial_t m_2 \, dx \, dt, \quad I_6 := \sum_i \int_0^T \int_{\mathbb{R}^3} (m_2 \times \partial_i m_2) \cdot \partial_i \partial_t m_1 \, dx \, dt,$$

$$I_7 := \int_0^T \int_{\mathbb{R}^3} (m_2 \times \partial_t m_2) \cdot \partial_t m_1 \, dx \, dt, \quad I_8 := \int_0^T \int_{\mathbb{R}^3} (m_1 \times \partial_t m_1) \cdot \partial_t m_2 \, dx \, dt.$$

Using Leibniz' rule and the properties of the triple product, we get

$$\begin{aligned}
I_1 + I_2 &= -\sum_i \int_0^T \int_{\mathbb{R}^3} (m_2 \times \partial_i \Delta m_1) \cdot \partial_i m_2 \, dx \, dt + \sum_i \int_0^T \int_{\mathbb{R}^3} (m_1 \times \partial_i \Delta m_1) \cdot \partial_i m_2 \, dx \, dt \\
&\quad + \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i m_1 \times \Delta m_1) \cdot \partial_i m_2 \, dx \, dt \\
&= \sum_i \int_0^T \int_{\mathbb{R}^3} (m \times \partial_i \Delta m_1) \cdot \partial_i m_2 \, dx \, dt - \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i m_1 \times \Delta m_1) \cdot \partial_i m \, dx \, dt \\
&= \sum_i \int_0^T \int_{\mathbb{R}^3} (m \times \partial_i \Delta m_1) \cdot \partial_i m_2 \, dx \, dt - \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i m \times \Delta m_1) \cdot \partial_i m \, dx \, dt \\
&= -\sum_i \int_0^T \int_{\mathbb{R}^3} (m \times \partial_i \Delta m_1) \cdot \partial_i m_2 \, dx \, dt + \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i (m \times \Delta m_1)) \cdot \partial_i m_1 \, dx \, dt \\
&\quad - \sum_i \int_0^T \int_{\mathbb{R}^3} (m \times \partial_i \Delta m_1) \cdot \partial_i m_1 \, dx \, dt.
\end{aligned}$$

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By an integration by parts, we see that the second term in the right hand side above vanishes. The remaining two terms can be combined into

$$I_1 + I_2 = -\sum_i \int_0^T \int_{\mathbb{R}^3} (m \times \partial_i \Delta m_1) \cdot \partial_i m \, dx \, dt. \tag{87}$$

We use again Leibniz' rule to expand I_3 into

$$I_3 = \sum_i \int_0^T \int_{\mathbb{R}^3} \left((\partial_i m_1) \times \partial_t m_1 \right) \cdot \partial_i m_2 \, dx \, dt + \sum_i \int_0^T \int_{\mathbb{R}^3} (m_1 \times \partial_i \partial_t m_1) \cdot \partial_i m_2 \, dx \, dt =: I_{3,a} + I_{3,b}. \tag{88}$$

Then, we observe that

$$I_{3,b} + I_6 = \sum_i \int_0^T \int_{\mathbb{R}^3} \det(m, \partial_i \partial_t m_1, \partial_i m_2) \, dx \, dt, \tag{89}$$

$$I_4 + I_5 = \int_0^T \int_{\mathbb{R}^3} \det(m, \partial_t m_2, \Delta m_1) \, dx \, dt. \tag{90}$$

On the other hand, we have

$$I_{3,a} = -\sum_i \int_0^T \int_{\mathbb{R}^3} \left((\partial_i m_1) \times \partial_t m_1 \right) \cdot \partial_i m \, dx \, dt$$

$$= \int_0^T \int_{\mathbb{R}^3} \left(\Delta m_1 \times \partial_t m_1 \right) \cdot m \, dx \, dt + \sum_i \int_0^T \int_{\mathbb{R}^3} \left((\partial_i m_1) \times \partial_t m_1 \right) \cdot m \, dx \, dt, \tag{91}$$

by integration by parts.

Combining (88)-(91) we obtain

$$I_3 + \ldots + I_6 = -\sum_i \int_0^T \int_{\mathbb{R}^3} \det(m, \partial_t \partial_i m_1, \partial_i m) \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} \det(m, \partial_t m, \Delta m_1) \, dx \, dt. \tag{92}$$

We have

$$I_7 + I_8 = \int_0^T \int_{\mathbb{R}^3} \left(\det(m, \partial_t m_2, \partial_t m_1) + \det(\partial_t m_2, m_1, \partial_t m_1) \right) dx \, dt$$

$$= -\int_0^T \int_{\mathbb{R}^3} \det(m, \partial_t m_2, \partial_t m_1) \, dx \, dt = \int_0^T \int_{\mathbb{R}^3} \det(m, \partial_t m, \partial_t m_1) \, dx \, dt. \tag{93}$$

Now, summing (87), (92) and (93), we deduce that

$$K_{\mathrm{LL}}[m_1, m_2] = 4 \sum_i \int_0^T \int_{\mathbb{R}^3} (m \times \partial_i \Delta m_1) \cdot \partial_i m \, dx \, dt - 2 \sum_i \int_0^T \int_{\mathbb{R}^3} (m \times \partial_t \partial_i m_1) \cdot \partial_i m \, dx \, dt \tag{94}$$

$$-2 \int_0^T \int_{\mathbb{R}^3} (m \times \partial_t m) \cdot \Delta m_1 \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} (m \times \partial_t m) \cdot \partial_t m_1 \, dx \, dt.$$

Since m vanishes at initial time, Poincaré's inequality yields

$$\int_0^T \int_{\mathbb{R}^3} |m|^2 \leqslant o(T) \int_0^T \int_{\mathbb{R}^3} |\partial_t m|^2 \, dx \, dt.$$

Thus, for T small enough, one gets

$$|K_{\mathrm{LL}}[m_1, m_2]| \leqslant \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} |\partial_t m|^2 \, dx \, dt + C \int_0^T \int_{\mathbb{R}^3} |\nabla m|^2 \, dx \, dt.$$

We combine this with inequality (81) and use Gronwall's lemma to conclude that m vanishes, first for small time, and the argument can be repeated as many times as necessary.

3.2 Case of the MLL equations

We consider $(m_2, E_2, H_2) : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^9$ a global weak solution to the MLL equations on $(0, \infty) \times \mathbb{R}^3$ satisfying the energy inequality, for almost every $T \geqslant 0$. Here the weak formulation reads: for every $\Psi \in H^1((0, T) \times \mathbb{R}^3; \mathbb{R}^3)$,

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t m_2 + m_2 \times \partial_t m_2) \cdot \Psi \, dx \, dt = -2 \sum_i \int_0^T \int_{\mathbb{R}^3} (m_2 \times \partial_i m_2) \cdot \partial_i \Psi \, dx \, dt \tag{95}$$

$$+ 2 \int_0^T \int_{\mathbb{R}^3} (m_2 \times H_2) \cdot \Psi \, dx \, dt,$$

$$- \int_0^T \int_{\mathbb{R}^3} (H_2 + m_2) \cdot \partial_t \Psi \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} E_2 \cdot \operatorname{curl} \Psi \, dx \, dt = \int_{\mathbb{R}^3} (H_0 + m_0) \cdot \Psi(0, \cdot) \, dx \tag{96}$$

$$- \Big(\int_{\mathbb{R}^3} (H_2 + m_2) \cdot \Psi \, dx \Big)(T),$$

$$- \int_0^T \int_{\mathbb{R}^3} E_2 \cdot \partial_t \Psi \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} H_2 \cdot \operatorname{curl} \Psi \, dx \, dt = \int_{\mathbb{R}^3} E_0 \cdot \Psi(0, \cdot) \, dx - \Big(\int_{\mathbb{R}^3} E_2 \cdot \Psi \, dx \Big)(T), \tag{97}$$

where the sum is over $1, 2, 3$. The initial data m_0 is prescribed in the trace sense, and is here assumed to be smooth.

Let us consider (m_1, E_1, H_1), regular solution to the MLL equations with the same initial data m_0. For $j = 1, 2$, let

$$\mathcal{E}_{\mathrm{MLL}}^j(t) := \int_{\mathbb{R}^3} \Big(|E_j|^2(t, x) + |H_j|^2(t, x) + |\nabla m_j|^2(t, x) \Big) dx \quad \text{and} \quad \mathcal{D}_{\mathrm{MLL}}^j(t) := \int_{\mathbb{R}^3} |\partial_t m_j|^2(t, x) \, dx.$$

Let us also introduce

$$J_{\mathrm{MLL}}^j(T) := \mathcal{E}_{\mathrm{MLL}}^j(T) + \int_0^T \mathcal{D}_{\mathrm{MLL}}^j \, dt$$

and

$$L_{\mathrm{MLL}}(T) := \int_{\mathbb{R}^3} \Big(|E|^2(T, x) + |H|^2(T, x) + |\nabla m|^2(T, x) \Big) dx + \int_0^T \int_{\mathbb{R}^3} |\partial_t m|^2(t, x) \, dx \, dt,$$

where $m := m_1 - m_2$, $E := E_1 - E_2$ and $H := H_1 - H_2$.

We first expand $L_{\mathrm{MLL}}(T)$ into

$$\begin{aligned} L_{\mathrm{MLL}}(T) &= J_{\mathrm{MLL}}^1(T) + J_{\mathrm{MLL}}^2(T) - 2 \Big(\int_{\mathbb{R}^3} \nabla m_1 : \nabla m_2 \, dx \Big)(T) - 2 \int_0^T \int_{\mathbb{R}^3} \partial_t m_1 \cdot \partial_t m_2 \, dx \, dt \\ &\quad - 2 \Big(\int_{\mathbb{R}^3} E_1 \cdot E_2 \, dx \Big)(T) - 2 \Big(\int_{\mathbb{R}^3} H_1 \cdot H_2 \, dx \Big)(T). \end{aligned}$$

Since the two solutions satisfy the energy inequality, and using (80), we get

$$L_{\mathrm{MLL}}(T) \leqslant 2 \tilde{L}_{\mathrm{MLL}}(T) + \tilde{K}_{\mathrm{MLL}}(T), \tag{98}$$

with

$$\begin{aligned} \tilde{K}_{\mathrm{MLL}}(T) &:= -2 \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i \partial_t m_1) \cdot \partial_i m_2 \, dx \, dt + 2 \sum_i \int_0^T \int_{\mathbb{R}^3} (\Delta m_1) \cdot \partial_t m_2 \, dx \, dt \\ &\quad - 2 \int_0^T \int_{\mathbb{R}^3} \partial_t m_1 \cdot \partial_t m_2 \, dx \, dt, \end{aligned}$$

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and

$$\tilde{L}_{\mathrm{MLL}}(T) := \int_{\mathbb{R}^3} \left(|E_0|^2(T,x) + |H_0|^2(T,x) \right) dx - \left(\int_{\mathbb{R}^3} E_1 \cdot E_2 \, dx \right)(T) - \left(\int_{\mathbb{R}^3} H_1 \cdot H_2 \, dx \right)(T).$$

Following the computations performed for the LL equations, taking into account the extra-term coming from the magnetic field in (95), we obtain:

$$\tilde{K}_{\mathrm{MLL}}(T) := K_{\mathrm{LL}}[m_1,m_2](T) - 2 \int_0^T \int_{\mathbb{R}^3} \det(m_2,H_2,\partial_t m_1) \, dx \, dt + 4 \int_0^T \int_{\mathbb{R}^3} \det(m_2,H_2,\Delta m_1) \, dx \, dt$$

$$-2 \int_0^T \int_{\mathbb{R}^3} \det(m_1,H_1,\partial_t m_2) \, dx \, dt + 4 \sum_i \int_0^T \int_{\mathbb{R}^3} \left(\partial_i(m_1 \times H_1) \right) \cdot \partial_i m_2 \, dx \, dt.$$

where $K_{\mathrm{LL}}[m_1,m_2](T)$ denotes here the right-hand-side of (94).

We use (96) and (97) respectively with $\Psi = H_1$ and $\Psi = E_1$ to get

$$\tilde{L}_{\mathrm{MLL}}(T) = -\int_0^T \int_{\mathbb{R}^3} (H_2 + m_2) \cdot \partial_t H_1 \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} E_2 \cdot \mathrm{curl}\, H_1 \, dx \, dt$$

$$-\int_{\mathbb{R}^3} H_0 \cdot m_0 \, dx + \int_{\mathbb{R}^3} m_2(T) \cdot H_1(T) \, dx$$

$$-\int_0^T \int_{\mathbb{R}^3} E_2 \cdot \partial_t E_1 \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} H_2 \cdot \mathrm{curl}\, E_1 \, dx \, dt,$$

and then, using that (m_1, E_1, H_1) satisfies Equations (2) and (3), we obtain

$$\tilde{L}_{\mathrm{MLL}}(T) = \int_0^T \int_{\mathbb{R}^3} H_2 \cdot \partial_t m_1 \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} H_1 \cdot \partial_t m_2 \, dx \, dt.$$

Now, we use on one hand that m_1 solves equation (1) and on the other hand equation (95) with $\Psi = H_1$ to obtain

$$\tilde{L}_{\mathrm{MLL}}(T) = -\int_0^T \int_{\mathbb{R}^3} \det(H_2,m_1,\partial_t m_1) \, dx \, dt + 2 \int_0^T \int_{\mathbb{R}^3} \det(H_2,m_1,\Delta m_1) \, dx \, dt$$

$$+2 \int_0^T \int_{\mathbb{R}^3} \det(H_2,m_1,H_1) \, dx \, dt$$

$$-\int_0^T \int_{\mathbb{R}^3} \det(H_1,m_2,\partial_t m_2) \, dx \, dt - 2 \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i H_1) \cdot (m_2 \times \partial_i m_2) \, dx \, dt$$

$$+2 \int_0^T \int_{\mathbb{R}^3} \det(H_1,m_2,H_2) \, dx \, dt.$$

Therefore

$$2\tilde{L}_{\mathrm{MLL}}(T) + \tilde{K}_{\mathrm{MLL}}(T) = K_{\mathrm{LL}}[m_1,m_2](T) + 2P_1(T) + 4P_2(T) + 4P_3(T), \tag{99}$$

where

$$P_1(T) := -\int_0^T \int_{\mathbb{R}^3} \det(m_2,H_2,\partial_t m_1) \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} \det(m_1,H_1,\partial_t m_2) \, dx \, dt$$

$$-\int_0^T \int_{\mathbb{R}^3} \det(H_2,m_1,\partial_t m_1) \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} \det(H_1,m_2,\partial_t m_2) \, dx \, dt,$$

$$P_2(T) \quad := \quad \sum_i \int_0^T \int_{\mathbb{R}^3} \left(\partial_i (m_1 \times H_1) \right) \cdot \partial_i m_2 \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \det(H_2, m_1, \Delta m_1) \, dx \, dt$$

$$- \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i H_1) \cdot (m_2 \times \partial_i m_2) \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \det(m_2, H_2, \Delta m_1) \, dx \, dt,$$

and

$$P_3(T) := \int_0^T \int_{\mathbb{R}^3} \det(H_1, m_2, H_2) \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \det(H_2, m_1, H_1) \, dx \, dt.$$

Now, we observe that

$$P_1(T) = - \int_0^T \int_{\mathbb{R}^3} \det(m, H, \partial_t m_1) \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \det(m, H_1, \partial_t m) \, dx \, dt, \tag{100}$$

and, by Leibniz' rule, that

$$P_2(T) \quad = \quad \sum_i \int_0^T \int_{\mathbb{R}^3} \det(\partial_i m_1, H_1, \partial_i m_2) \, dx \, dt + \sum_i \int_0^T \int_{\mathbb{R}^3} \det(m_1, \partial_i H_1, \partial_i m_2) \, dx \, dt$$

$$+ \int_0^T \int_{\mathbb{R}^3} \det(H_2, m, \Delta m_1) \, dx \, dt - \sum_i \int_0^T \int_{\mathbb{R}^3} \det(\partial_i H_1, m_2, \partial_i m_2) \, dx \, dt.$$

Now, we use an integration by parts to get that

$$\int_0^T \int_{\mathbb{R}^3} \det(\partial_i m_1, H_1, \partial_i m_2) \, dx \, dt \quad = \quad \int_0^T \int_{\mathbb{R}^3} \det(\partial_i m_1, H_1, \partial_i m) \, dx \, dt$$

$$= \quad - \int_0^T \int_{\mathbb{R}^3} \det(\partial_i^2 m_1, H_1, m) \, dx \, dt$$

$$- \int_0^T \int_{\mathbb{R}^3} \det(\partial_i m_1, \partial_i H_1, m) \, dx \, dt.$$

Thus

$$P_2(T) = - \int_0^T \int_{\mathbb{R}^3} \det(\Delta m_1, H, m) \, dx \, dt - \sum_i \int_0^T \int_{\mathbb{R}^3} \det(\partial_i m, \partial_i H_1, m) \, dx \, dt. \tag{101}$$

Finally, we easily get

$$P_3(T) = \int_0^T \int_{\mathbb{R}^3} \det(H_1, m, H) \, dx \, dt. \tag{102}$$

Plugging (99)-(100)-(101)-(102) into (98) one gets

$$L_{\mathrm{MLL}}(T) \leqslant \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} |\partial_t m|^2 \, dx \, dt + C \int_0^T \int_{\mathbb{R}^3} \left(|E|^2 + |H|^2 + |\nabla m|^2 \right) dx \, dt,$$

and using a Gronwall lemma yields the desired conclusion.

4 Weak-strong uniqueness: Proof of Theorem 5

We will prove Theorem 5 in a simplified setting which focuses on the difficulty due to the Hall effect. The extension to the general case is straightforward. Therefore we consider the following equations:

$$\partial_t B + \operatorname{curl}\big((\operatorname{curl} B \times B)\big) = \Delta B, \tag{103}$$

$$\operatorname{div} B = 0. \tag{104}$$

We consider a global weak solution B_2 to (103)-(104) associated with an initial data $B_0 \in \mathcal{H}$, assumed smooth.

Here the weak formulation reads: for any $\Psi \in C^1([0,T]; C_c^1(\mathbb{R}^3))$, for any $T > 0$,

$$-\int_0^T \int_{\mathbb{R}^3} (\partial_t \Psi) \cdot B_2 \, dx \, dt + \big(\int_{\mathbb{R}^3} \Psi \cdot B_2\big)|_{t=T} \, dx - \big(\int_{\mathbb{R}^3} \Psi|_{t=0} \cdot B_0 \, dx\big) \tag{105}$$

$$+ \int_0^T \int_{\mathbb{R}^3} (\operatorname{curl} \Psi) \cdot \big((\operatorname{curl} B_2) \times B_2\big) \, dx \, dt = -\int_0^T \int_{\mathbb{R}^3} (\operatorname{curl} \Psi) \cdot (\operatorname{curl} B_2) \, dx \, dt,$$

and the energy inequality: for almost every $T > 0$,

$$J_{\mathrm{HMHD}}[B_2](T) := \frac{1}{2}\big(\int_{\mathbb{R}^3} B_2 \, dx\big)|_{t=T} + \int_0^T \int_{\mathbb{R}^3} (\operatorname{curl} B_2)^2 \, dx \, dt \leqslant \frac{1}{2}\int_{\mathbb{R}^3} B_0^2 \, dx. \tag{106}$$

We also consider a regular solution B_2 of (103)-(103) on $(0, T_0)$, for $T_0 > 0$. We denote $B := B_1 - B_2$ and expand $J_{\mathrm{HMHD}}[B](T)$ into

$$J_{\mathrm{HMHD}}[B](T) = J_{\mathrm{HMHD}}[B_1](T) + J_{\mathrm{HMHD}}[B_2](T) - \big(\int_{\mathbb{R}^3} B_1 \cdot B_2 \, dx\big)(T) - 2\int_0^T \int_{\mathbb{R}^3} (\operatorname{curl} B_1) \cdot (\operatorname{curl} B_2) \, dx \, dt,$$

and then we use that both B_1 and B_2 satisfy the weak energy inequality (106) to deduce that

$$J_{\mathrm{HMHD}}[B](T) \leqslant \int_{\mathbb{R}^3} B_0^2 \, dx - \big(\int_{\mathbb{R}^3} B_1 \cdot B_2 \, dx\big)(T) - 2\int_0^T \int_{\mathbb{R}^3} (\operatorname{curl} B_1) \cdot (\operatorname{curl} B_2) \, dx \, dt. \tag{107}$$

We use (105) with $\Psi = B_1$ to get

$$-\int_{\mathbb{R}^3} B_0^2 + \big(\int_{\mathbb{R}^3} B_1 \cdot B_2 \, dx\big)(T) + \int_0^T \int_{\mathbb{R}^3} (\operatorname{curl} B_1) \cdot (\operatorname{curl} B_2) \, dx \, dt = \int_0^T \int_{\mathbb{R}^3} (\partial_t B_1) \cdot B_2 \, dx \, dt \tag{108}$$

$$- \int_0^T \int_{\mathbb{R}^3} (\operatorname{curl} B_1) \cdot \big((\operatorname{curl} B_2) \times B_2\big) \, dx \, dt.$$

Now we use that B_1 satisfies (103)-(104) to obtain

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t B_1) \cdot B_2 \, dx \, dt = -\int_0^T \int_{\mathbb{R}^3} \big((\operatorname{curl} B_1) \times B_1)\big) \cdot \operatorname{curl} B_2 \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} (\operatorname{curl} B_1) \cdot (\operatorname{curl} B_2) \, dx \, dt.$$

Plugging this into (108) provides that

$$L := -\int_{\mathbb{R}^3} B_0^2 + \big(\int_{\mathbb{R}^3} B_1 \cdot B_2\big)(T) \, dx + 2\int_0^T \int_{\mathbb{R}^3} (\operatorname{curl} B_1) \cdot (\operatorname{curl} B_2) \, dx \, dt$$

is given by

$$
\begin{aligned}
L &= -\int_0^T \int_{\mathbb{R}^3} \left((\operatorname{curl} B_1) \times B_1 \right) \cdot \operatorname{curl} B_2 \, dx \, dt - \int_0^T \int_{\mathbb{R}^3} (\operatorname{curl} B_1) \cdot (\operatorname{curl} B_2 \times B_2) \, dx \, dt \\
&= -\int_0^T \int_{\mathbb{R}^3} \det((\operatorname{curl} B_1), B_1, \operatorname{curl} B_2) \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \det((\operatorname{curl} B_1), B_2, \operatorname{curl} B_2) \, dx \, dt \\
&= -\int_0^T \int_{\mathbb{R}^3} \det((\operatorname{curl} B_1), B, \operatorname{curl} B_2) \, dx \, dt \\
&= -\int_0^T \int_{\mathbb{R}^3} \det((\operatorname{curl} B_1), B, \operatorname{curl} B) \, dx \, dt,
\end{aligned}
$$

and therefore combining with (107) we get

$$
\begin{aligned}
J_{\mathrm{HMHD}}[B](T) &\leqslant \int_0^T \int_{\mathbb{R}^3} \det((\operatorname{curl} B_1), B, \operatorname{curl} B) \, dx \, dt \\
&\leqslant C \int_0^T \int_{\mathbb{R}^3} B^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} (\operatorname{curl} B)^2 \, dx \, dt,
\end{aligned}
$$

which leads to the conclusion, thanks to a Gronwall estimate.

5 Local conservations: Proof of Part i) of Theorem 7, Theorem 8 and Theorem 9

We use repetitively in the sequel the following formula: for two smooth vector fields v and w there holds

$$
- v \cdot \operatorname{curl} w + w \cdot \operatorname{curl} v = \operatorname{div}(v \times w). \tag{109}
$$

5.1 MLL equations: Proof of Part i) of Theorem 7

We first take the convolution of the equations with the mollifier ψ^ε in order to obtain the regularized equations:

$$
\partial_t m_\varepsilon + (m \times \partial_t m)_\varepsilon = 2 \sum_i \partial_i \left(m \times \partial_i m \right)_\varepsilon + 2(m \times H)_\varepsilon, \tag{110}
$$

$$
\partial_t H_\varepsilon + \operatorname{curl} E_\varepsilon = -\partial_t m_\varepsilon, \tag{111}
$$

$$
\partial_t E_\varepsilon - \operatorname{curl} H_\varepsilon = 0, \tag{112}
$$

$$
\operatorname{div} E_\varepsilon = \operatorname{div}(H_\varepsilon + m_\varepsilon) = 0, \tag{113}
$$

where we use notation (53).

Next we apply formula (55), so that (110) becomes

$$
\partial_t m_\varepsilon + m_\varepsilon \times \partial_t m_\varepsilon + \mathcal{B}^\varepsilon[m, \partial_t m] = 2 m_\varepsilon \times \Delta m_\varepsilon + 2\mathcal{B}^\varepsilon[m, \Delta m] + 2 m_\varepsilon \times H_\varepsilon + 2\mathcal{B}^\varepsilon[m, H]. \tag{114}
$$

Let us take the inner product of (114) with $\partial_t m_\varepsilon$, Δm_ε and H_ε to get

$$
\begin{aligned}
|\partial_t m_\varepsilon|^2 + \mathcal{B}^\varepsilon[m, \partial_t m] \cdot \partial_t m_\varepsilon &= 2(m_\varepsilon \times \Delta m_\varepsilon) \cdot \partial_t m_\varepsilon + 2\mathcal{B}^\varepsilon[m, \Delta m] \cdot \partial_t m_\varepsilon \\
&\quad + 2(m_\varepsilon \times H_\varepsilon) \cdot \partial_t m_\varepsilon + 2\mathcal{B}^\varepsilon[m, H] \cdot \partial_t m_\varepsilon,
\end{aligned} \tag{115}
$$

$$\partial_t m_\varepsilon \cdot \Delta m_\varepsilon + (m_\varepsilon \times \partial_t m_\varepsilon) \cdot \Delta m_\varepsilon + \mathcal{B}^\varepsilon[m, \partial_t m] \cdot \Delta m_\varepsilon = 2\mathcal{B}^\varepsilon[m, \Delta m] \cdot \Delta m_\varepsilon \qquad (116)$$
$$+ 2(m_\varepsilon \times H_\varepsilon) \cdot \Delta m_\varepsilon + 2\mathcal{B}^\varepsilon[m, H] \cdot \Delta m_\varepsilon,$$

$$\partial_t m_\varepsilon \cdot H_\varepsilon + (m_\varepsilon \times \partial_t m_\varepsilon) \cdot H_\varepsilon + \mathcal{B}^\varepsilon[m, \partial_t m] \cdot H_\varepsilon = 2(m_\varepsilon \times \Delta m_\varepsilon) \cdot H_\varepsilon + 2\mathcal{B}^\varepsilon[m, \Delta m] \cdot H_\varepsilon \quad (117)$$
$$+ 2(m_\varepsilon \times H_\varepsilon) \cdot H_\varepsilon + 2\mathcal{B}^\varepsilon[m, H] \cdot H_\varepsilon.$$

On the other hand we take the inner product of (111) with H_ε, the inner product of (112) with E_ε and we take the sum to get

$$\partial_t(|E_\varepsilon|^2 + |H_\varepsilon|^2) + 2\operatorname{div}(H_\varepsilon \times E_\varepsilon) = -2H_\varepsilon \cdot \partial_t m_\varepsilon. \qquad (118)$$

Let us now compute $(115) - 2(116) - 2(117) + (118)$. This yields

$$|\partial_t m_\varepsilon|^2 - 2\partial_t m_\varepsilon \cdot \Delta m_\varepsilon + \partial_t(|E_\varepsilon|^2 + |H_\varepsilon|^2) + 2\operatorname{div}(H_\varepsilon \times E_\varepsilon) = d_{\mathrm{MLL}}^{\mathfrak{a},\varepsilon},$$

with $d_{\mathrm{MLL}}^{\mathfrak{a},\varepsilon}$ given by (58).

Therefore it suffices to observe that

$$-2\partial_t m \cdot \Delta m = -2\sum_i \partial_i\Big(\partial_t m \cdot \partial_i m\Big) + \partial_t(|\nabla m|^2),$$

to obtain

$$\partial_t e_{\mathrm{MLL}}^\varepsilon + d_{\mathrm{MLL}}^\varepsilon + \operatorname{div} f_{\mathrm{MLL}}^\varepsilon = d_{\mathrm{MLL}}^{\mathfrak{a},\varepsilon}, \qquad (119)$$

where

$$e_{\mathrm{MLL}}^\varepsilon := |E_\varepsilon|^2 + |H_\varepsilon|^2 + |\nabla m_\varepsilon|^2, \quad d_{\mathrm{MLL}}^\varepsilon := |\partial_t m_\varepsilon|^2 \quad f_{\mathrm{MLL}}^\varepsilon := -2(\partial_t m_\varepsilon \cdot \partial_i m_\varepsilon)_{i=1,2,3} + 2H_\varepsilon \times E_\varepsilon. \quad (120)$$

Now we prove that, when $\varepsilon \to 0$, $d_{\mathrm{MLL}}^{\mathfrak{a},\varepsilon}$ converges, in the sense of distributions, to $d_{\mathrm{MLL}}^{\mathfrak{a}}$ whatever is the choice of the mollifier. Let us recall that $d_{\mathrm{MLL}}^{\mathfrak{a}}$ is given by (57).

Indeed it follows from the regularity of m, E and H that $|\partial_t m_\varepsilon|^2$, $|E_\varepsilon|^2$, $|H_\varepsilon|^2$, $|\nabla m_\varepsilon|^2$, $\partial_t m_\varepsilon \cdot \partial_i m_\varepsilon$ and $H_\varepsilon \times E_\varepsilon$ converge respectively to $|\partial_t m|^2$, $|E|^2$, $|H|^2$, $|\nabla m|^2$, $\partial_t m \cdot \partial_i m$ and $H \times E$ in $L^1((0,T) \times \mathbb{R}^3)_{\mathrm{loc}}$. As a consequence the left hand side of (119) converges, in the sense of distributions, to

$$|\partial_t m|^2 + \partial_t(|E|^2 + |H|^2 + |\nabla m|^2) - 2\sum_i \partial_i\Big(\partial_t m \cdot \partial_i m\Big) + 2\operatorname{div}(H \times E).$$

This entails that, $d_{\mathrm{MLL}}^{\mathfrak{a},\varepsilon}$ converges, in the sense of distributions, to $d_{\mathrm{MLL}}^{\mathfrak{a}}$.

5.2 HMHD equations: Proof of Part i) of Theorem 8 and Theorem 9

5.2.1 Regularization

We start with the regularized equations:

$$\partial_t u_\varepsilon + \operatorname{div}(u \otimes u - B \otimes B)_\varepsilon + \nabla(p_m)_\varepsilon = \Delta u_\varepsilon, \qquad (121)$$
$$\operatorname{div} u_\varepsilon = 0, \qquad (122)$$
$$\partial_t B_\varepsilon - \operatorname{curl}(u \times B)_\varepsilon + \operatorname{curl}\operatorname{div}(B \otimes B)_\varepsilon = \Delta B_\varepsilon, \qquad (123)$$
$$\operatorname{div} B_\varepsilon = 0. \qquad (124)$$

We use the decompositions (54), (55) and (56) to recast (121) and (123) as follows:

$$\partial_t u_\varepsilon + \operatorname{div}(u_\varepsilon \otimes u_\varepsilon - B_\varepsilon \otimes B_\varepsilon) + \operatorname{div}(\mathcal{C}^\varepsilon[u,u] - \mathcal{C}^\varepsilon[B,B]) + \nabla(p_\varepsilon + \frac{1}{2}|B_\varepsilon|^2) + \frac{1}{2}\nabla\mathcal{A}^\varepsilon[B,B] = \Delta u_\varepsilon, \quad (125)$$

$$\partial_t B_\varepsilon - \operatorname{curl}(u_\varepsilon \times B_\varepsilon) - \operatorname{curl}\mathcal{B}^\varepsilon[u,B] + \operatorname{curl}\Big((\operatorname{curl} B_\varepsilon) \times B_\varepsilon + \operatorname{div}\mathcal{C}^\varepsilon[B,B]\Big) = \Delta B_\varepsilon. \quad (126)$$

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5.2.2 Local magneto-helicity identity: Proof of Part i) of Theorem 8

Thanks to Leibniz' identity and (109) there holds

$$\partial_t(A_\varepsilon \cdot B_\varepsilon) = (\partial_t A_\varepsilon) \cdot B_\varepsilon + A_\varepsilon \cdot \partial_t B_\varepsilon = 2A_\varepsilon \cdot \partial_t B_\varepsilon + \mathrm{div}(A_\varepsilon \times \partial_t A_\varepsilon).$$

Using now (126) we obtain

$$\begin{aligned}
\partial_t(A_\varepsilon \cdot B_\varepsilon) &= -2A_\varepsilon \cdot \mathrm{curl}\Big((\mathrm{curl}\, B_\varepsilon - u_\varepsilon) \times B_\varepsilon)\Big) + 2A_\varepsilon \cdot \Delta B_\varepsilon + \mathrm{div}(A_\varepsilon \times \partial_t A_\varepsilon) \\
&\quad + 2A_\varepsilon \cdot \mathrm{curl}\, \mathcal{B}^\varepsilon[u, B] - 2A_\varepsilon \cdot \mathrm{curl}\, \mathrm{div}\, \mathcal{C}^\varepsilon[B, B].
\end{aligned}$$

Thanks to (109) and to the divergence free conditions we obtain

$$-A_\varepsilon \cdot \mathrm{curl}\Big((\mathrm{curl}\, B_\varepsilon - u_\varepsilon) \times B_\varepsilon)\Big) = \mathrm{div}\Big(A_\varepsilon \times ((\mathrm{curl}\, B_\varepsilon - u_\varepsilon) \times B_\varepsilon)\Big),$$

and

$$A_\varepsilon \cdot \Delta B_\varepsilon = -A_\varepsilon \cdot \mathrm{curl}\, \mathrm{curl}\, B_\varepsilon = -B_\varepsilon \cdot \mathrm{curl}\, B_\varepsilon + \mathrm{div}(A_\varepsilon \times \mathrm{curl}\, B_\varepsilon).$$

Therefore

$$\partial_t(A_\varepsilon \cdot B_\varepsilon) + 2B_\varepsilon \cdot \mathrm{curl}\, B_\varepsilon - \mathrm{div}\Big(2\big((u_\varepsilon - \mathrm{curl}\, B_\varepsilon) \times B_\varepsilon - 2\, \mathrm{curl}\, B_\varepsilon - \partial_t A_\varepsilon\big) \times A_\varepsilon\Big) = d_m^{\mathfrak{a},\varepsilon}, \qquad (127)$$

where $d_m^{\mathfrak{a},\varepsilon}$ is given by (64).

Let us now prove that $d_m^{\mathfrak{a},\varepsilon}$ converges in the sense of distributions to $d_m^{\mathfrak{a}}$. Indeed we are going to prove that the left hand side of (127) converges to

$$\partial_t(A \cdot B) + 2B \cdot \mathrm{curl}\, B - \mathrm{div}\Big(2(u - \mathrm{curl}\, B) \times B - 2\, \mathrm{curl}\, B - \partial_t A) \times A\Big).$$

Actually thanks to the estimates given by the existence theorem 2, and the fact that the vector potential A is in $L^2(0, T; H^2(\mathbb{R}^3))$ by elliptic regularity, we easily infer that

$$\partial_t(A_\varepsilon \cdot B_\varepsilon) + 2B_\varepsilon \cdot \mathrm{curl}\, B_\varepsilon - \mathrm{div}\Big((2(u_\varepsilon - \mathrm{curl}\, B_\varepsilon) \times B_\varepsilon - 2\, \mathrm{curl}\, B_\varepsilon) \times A_\varepsilon\Big)$$

converges in the sense of distributions to

$$\partial_t(A \cdot B) + 2B \cdot \mathrm{curl}\, B - \mathrm{div}\Big((2(u - \mathrm{curl}\, B) \times B - 2\, \mathrm{curl}\, B) \times A\Big).$$

Let us now turn our attention to the last term of the left hand side of (127). Using again elliptic regularity, we easily infer that A is in $L^\infty(0, T; \mathcal{V})$. Moreover using equation (7), one infers that $\partial_t A$ is in $L^{\frac{4}{3}}(0, T; \mathcal{V}')$. Actually this estimate is used in course of proving the existence theorem 2, see [1]. From that we deduce that $\mathrm{div}\Big((\partial_t A_\varepsilon) \times A_\varepsilon\Big)$ converges in the sense of distributions to $\mathrm{div}\Big((\partial_t A) \times A\Big)$.

This concludes the proof of the first part of Theorem 8.

5.2.3 Local Energy identity: Proof of Part i) of Theorem 9

Let us take the inner product of (125) and (126) respectively with u_ε and B_ε, and sum the resulting identities, taking into account that, thanks to (122) and (124),

$$u_\varepsilon \cdot \mathrm{div}(u_\varepsilon \otimes u_\varepsilon) + u_\varepsilon \cdot \nabla p_\varepsilon = \mathrm{div}\Big((\tfrac{1}{2}|u_\varepsilon|^2 + p_\varepsilon)u_\varepsilon\Big),$$

$$u_\varepsilon \cdot \Big(-\operatorname{div}(B_\varepsilon \otimes B_\varepsilon) + \frac{1}{2}\nabla(|B_\varepsilon|^2) \Big) = -u_\varepsilon \cdot \Big((\operatorname{curl} B_\varepsilon) \times B_\varepsilon \Big) = (\operatorname{curl} B_\varepsilon) \cdot \Big(u_\varepsilon \times B_\varepsilon \Big),$$

so that

$$u_\varepsilon \cdot \Big(-\operatorname{div}(B_\varepsilon \otimes B_\varepsilon) + \frac{1}{2}\nabla(|B_\varepsilon|^2) \Big) - B_\varepsilon \cdot \Big(\operatorname{curl}(u_\varepsilon \times B_\varepsilon) \Big) = \operatorname{div}\Big(B_\varepsilon \times (u_\varepsilon \times B_\varepsilon) \Big),$$

$$B_\varepsilon \cdot \Big(\operatorname{curl}\Big((\operatorname{curl} B_\varepsilon) \times B_\varepsilon \Big) \Big) = \operatorname{div}\Big(\big((\operatorname{curl} B_\varepsilon) \times B_\varepsilon \big) \times B_\varepsilon \Big),$$

and

$$-u_\varepsilon \cdot \Delta u_\varepsilon - B_\varepsilon \cdot \Delta B_\varepsilon = |\operatorname{curl} u_\varepsilon|^2 + |\operatorname{curl} B_\varepsilon|^2 + \operatorname{div}\Big((\operatorname{curl} u_\varepsilon) \times u_\varepsilon + (\operatorname{curl} B_\varepsilon) \times B_\varepsilon \Big).$$

We thus obtain

$$\partial_t e^\varepsilon_{\mathrm{HMHD}} + d^\varepsilon_{\mathrm{HMHD}} + \operatorname{div} f^\varepsilon_{\mathrm{HMHD}} = d^{\mathfrak{a},\varepsilon}_{\mathrm{HMHD}}, \tag{128}$$

where

$$e^\varepsilon_{\mathrm{HMHD}} := \frac{1}{2}\Big(|u_\varepsilon|^2 + |B_\varepsilon|^2 \Big), \quad d^\varepsilon_{\mathrm{HMHD}} := |\operatorname{curl} u_\varepsilon|^2 + |\operatorname{curl} B_\varepsilon|^2, \tag{129}$$

$$f^\varepsilon_{\mathrm{HMHD}} := (\frac{1}{2}|u_\varepsilon|^2 + p_\varepsilon)u_\varepsilon + B_\varepsilon \times (u_\varepsilon \times B_\varepsilon) + (\operatorname{curl} u_\varepsilon) \times u_\varepsilon + (\operatorname{curl} B_\varepsilon) \times B_\varepsilon + \big((\operatorname{curl} B_\varepsilon) \times B_\varepsilon \big) \times B_\varepsilon,$$

$$d^{\mathfrak{a},\varepsilon}_{\mathrm{HMHD}} := -u_\varepsilon \cdot \operatorname{div}\Big(\mathcal{C}^\varepsilon[u,u] - \mathcal{C}^\varepsilon[B,B] \Big) - \frac{1}{2}u_\varepsilon \cdot \nabla \mathcal{A}^\varepsilon[B,B] + B_\varepsilon \cdot \operatorname{curl} \mathcal{B}^\varepsilon[u,B] - B_\varepsilon \cdot \operatorname{curl}\operatorname{div}\mathcal{C}^\varepsilon[B,B].$$

Let us now prove that if u and B are given by Theorem 2 with $B \in L^4((0,T) \times \mathbb{R}^3)_{\mathrm{loc}}$ then $d^{\mathfrak{a},\varepsilon}_{\mathrm{HMHD}}$ converges, in the sense of distributions, to $d^{\mathfrak{a}}_{\mathrm{HMHD}}$.

First observe that since u and B belong to $L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$, they also belong to $L^3((0,T) \times \mathbb{R}^3)$.

It is not difficult to see that, under these assumptions,

$$|u_\varepsilon|^2 + |B_\varepsilon|^2, |u_\varepsilon|^2 u_\varepsilon, B_\varepsilon \times (u_\varepsilon \times B_\varepsilon), (\operatorname{curl} u_\varepsilon) \times u_\varepsilon, (\operatorname{curl} B_\varepsilon) \times B_\varepsilon, \big((\operatorname{curl} B_\varepsilon) \times B_\varepsilon \big) \times B_\varepsilon,$$
$$|\operatorname{curl} u_\varepsilon|^2 \text{ and } |\operatorname{curl} B_\varepsilon|^2$$

converge in $L^1((0,T) \times \mathbb{R}^3)_{\mathrm{loc}}$ respectively to

$$|u|^2 + |B|^2, |u|^2 u, B \times (u \times B), (\operatorname{curl} u) \times u, (\operatorname{curl} B) \times B, \big((\operatorname{curl} B) \times B \big) \times B, |\operatorname{curl} u|^2 \text{ and } |\operatorname{curl} B|^2.$$

Therefore in order to prove that the left hand side of (128) converges in the sense of distributions to the left hand side of (38) it is sufficient to prove that p_ε converges in $L^{\frac{3}{2}}((0,T) \times \mathbb{R}^3)_{\mathrm{loc}}$ to p. But taking the divergence of (28), and taking (29) into account, we obtain that the magnetic pressure p_m satisfies

$$\Delta p_m = -\operatorname{div}\operatorname{div}(u \otimes u - B \otimes B). \tag{130}$$

Then classical elliptic regularity allows to conclude that p_m, and therefore p, is in $L^{\frac{3}{2}}((0,T) \times \mathbb{R}^3)_{\mathrm{loc}}$, what concludes the proof of Part i) of Theorem 9.

6 Technicalities

In this section we gather a few technical results which will be useful in the sequel.

6.1 Some injections

The following lemma is a consequence of Bernstein's lemma. Its proof is given in the Appendix.

Lemma 14. *Let $\alpha, \tilde{\alpha} \in (0,1) \cup (1,2)$, and $p, r \in [1, \infty]$. Assume that $\tilde{\alpha} \leqslant \alpha$ and define $\tilde{p} \in [1, \infty]$ by $\tilde{\alpha} - 3/\tilde{p} = \alpha - 3/p$. Then $\widetilde{L}^r(0, T; \dot{B}_{p,\infty}^{\alpha}(\mathbb{R}^3))$ is continuously embedded in $\widetilde{L}^r(0, T; \dot{B}_{\tilde{p},\infty}^{\tilde{\alpha}}(\mathbb{R}^3))$. As a consequence, any $u \in \widetilde{L}^r(0, T; \dot{B}_{p,\infty}^{\alpha}(\mathbb{R}^3))_{\mathrm{loc}}$ belongs to $\widetilde{L}^r(0, T; \dot{B}_{\tilde{p},\infty}^{\tilde{\alpha}}(\mathbb{R}^3))_{\mathrm{loc}}$.*

Let us observe that if (p, α) satisfies the relationship in (59) then so does any $(\tilde{p}, \tilde{\alpha})$ such that $\tilde{\alpha} - 3/\tilde{p} = \alpha - 3/p$. Similarly if (q, β) satisfies $q = 12/(4\beta - 1)$ (see Remark 3), then so does any $(\tilde{q}, \tilde{\beta})$ such that $\tilde{\beta} - 3/\tilde{q} = \beta - 3/q$.

We will use in particular that

- for $\alpha \in [4/3, 11/6]$ and $p := \dfrac{9}{3\alpha - 1}$, a function belonging to the space $\widetilde{L}^3(0, T; \dot{B}_{p,\infty}^{\alpha}(\mathbb{R}^3))_{\mathrm{loc}}$ also belongs to the space $\widetilde{L}^3(0, T; \dot{B}_{\tilde{p},\infty}^{\tilde{\alpha}}(\mathbb{R}^3))_{\mathrm{loc}}$, with $\tilde{\alpha} := 4 - 2\alpha$ and $\tilde{p} := \dfrac{9}{3\tilde{\alpha} - 1}$ which satisfies $\dfrac{2}{p} + \dfrac{1}{\tilde{p}} = 1$.

- for $\beta \in (9/8, 3/2)$ and $q := \dfrac{12}{4\beta - 1}$, a function belonging to the space $\widetilde{L}^4(0, T; \dot{B}_{q,\infty}^{\beta}(\mathbb{R}^3))_{\mathrm{loc}}$ also belongs to the space $\widetilde{L}^4(0, T; \dot{B}_{\tilde{q},\infty}^{2-\beta}(\mathbb{R}^3))_{\mathrm{loc}}$, with $\tilde{q} := \dfrac{3}{\frac{7}{4} - \beta}$ which satisfies $\dfrac{1}{q} + \dfrac{1}{\tilde{q}} = \dfrac{1}{2}$.

6.2 A Constantin-E-Titi type lemma

We will make a crucial use of the following lemma adapted from [21]. The notations $\mathcal{A}^\varepsilon, \mathcal{B}^\varepsilon, \mathcal{C}^\varepsilon$ are from (54), (55), (56).

Lemma 15. *Let $i = 0$, 1 or 2. Let $(r_1, r_2, r_3) \in [1, \infty]^3$, $(p_1, p_2, p_3) \in [1, \infty]^3$ and $(\alpha_1, \alpha_2, \alpha_3) \in [0,1)^3$ be such that*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \quad and \quad \alpha_1 + \alpha_2 + \alpha_3 \geqslant i.$$

Let ϕ^1, ϕ^2, ϕ^3 be functions on $(0, T) \times \mathbb{R}^3$ such that for $j = 1, 2, 3$,

- *either $\alpha_j \in (0,1)$ and $\phi^j \in \widetilde{L}^{r_j}(0, T; \dot{B}_{p_j,\infty}^{\alpha_j}(\mathbb{R}^3))_{\mathrm{loc}}$,*

- *or $\alpha_j = 0$, and $\phi^j \in L^{r_j}(0, T; L^{p_j}(\mathbb{R}^3))_{\mathrm{loc}}$,*

and such that for at least one index $j \in \{1, 2, 3\}$ ($j \in \{1, 2\}$ in the case where $\alpha_1 + \alpha_2 + \alpha_3 = i = 0$),

- *either $\alpha_j \in (0,1)$ and $\phi^j \in \widetilde{L}^{r_j}(0, T; \dot{B}_{p_j,c_0}^{\alpha_j}(\mathbb{R}^3))_{\mathrm{loc}}$,*

- *or $\alpha_j = 0$, $r_j, p_j < \infty$ and $\phi^j \in L^{r_j}(0, T; L^{p_j}(\mathbb{R}^3))_{\mathrm{loc}}$.*

Then, for all $\chi \in C_{\mathrm{c}}^{\infty}((0, T) \times \mathbb{R}^3)$,

$$\int_0^T \int_{\mathbb{R}^3} \chi \Big(|\mathcal{A}^\varepsilon[\phi^1, \phi^2]| + |\mathcal{B}^\varepsilon[\phi^1, \phi^2]| + |\mathcal{C}^\varepsilon[\phi^1, \phi^2]| \Big) |\nabla^i \phi_\varepsilon^3| \, dx \, dt \to 0 \quad when \ \varepsilon \to 0. \tag{131}$$

Before proving Lemma 15, we start with a few preliminary results.

Lemma 16. *Let $\alpha \in (0,1)$, and $p, r \in [1, \infty]$.*

1. For all $u \in \widetilde{L}^r(0, T; \dot{B}_{p,\infty}^{\alpha}(\mathbb{R}^3))$,

$$\|u - u_\varepsilon\|_{L^r((0,T); L^p(\mathbb{R}^3))} = O(\varepsilon^\alpha). \tag{132}$$

2. For all $u \in \widetilde{L}^r(0,T; \dot{B}_{p,c_0}^\alpha(\mathbb{R}^3))$,

$$\|u - u_\varepsilon\|_{L^r((0,T);L^p(\mathbb{R}^3))} = o(\varepsilon^\alpha). \tag{133}$$

3. If $r, p \in [1, \infty)$, then for all $u \in L^r(0,T; L^p(\mathbb{R}^3))$,

$$\|u - u_\varepsilon\|_{L^r((0,T);L^p(\mathbb{R}^3))} = o(1). \tag{134}$$

Proof. We begin with the proof of (132). We use that

$$u_\varepsilon(t,x) - u(t,x) = \int_{\mathbb{R}^3} \psi^\varepsilon(y)(u(t,x-y) - u(t,x))dy,$$

so that, with $\delta_y u(t,x) = u(t,x-y) - u(t,x)$,

$$\|u - u_\varepsilon\|_{L^r(0,T;L^p(\mathbb{R}^3))} \leqslant \int_{\mathbb{R}^3} \psi^\varepsilon(y)\|\delta_y u\|_{L^r(0,T;L^p(\mathbb{R}^3))}dy. \tag{135}$$

Now, this implies

$$\|u - u_\varepsilon\|_{L^r(0,T;L^p(\mathbb{R}^3))} \leqslant \left(\int_{\mathbb{R}^3} \psi^\varepsilon(y)|y|^\alpha dy\right) \|u\|_{\widetilde{L}^r(0,T;B_{p,\infty}^\alpha(\mathbb{R}^3))}.$$

According to the size of the support of ψ^ε, this proves (132). To obtain (133), we write

$$\varepsilon^{-\alpha}\|u - u_\varepsilon\|_{L^r(0,T;L^p(\mathbb{R}^3))} \leqslant \int_{\mathbb{R}^3} \psi^\varepsilon(y)\left(\frac{|y|}{\varepsilon}\right)^\alpha \|f_{\alpha,p}[u](y)\|_{L^r(0,T)}dy \leqslant C\int_{\mathbb{R}^3} \psi^\varepsilon(y)\|f_{\alpha,p}[u](y)\|_{L^r(0,T)}dy,$$

and again, the fact that $\text{supp}(\psi^\varepsilon)$ has size ε yields the result.

To prove (134), we come back to (135). For all $t \in (0,T)$, $\|\delta_y u(t,\cdot)\|_{L^p(\mathbb{R}^3)} \leqslant 2\|u(t,\cdot)\|_{L^p(\mathbb{R}^3)}$, and since $p < \infty$, $\|\delta_y u(t,\cdot)\|_{L^p(\mathbb{R}^3)} \underset{y \to 0}{\longrightarrow} 0$, so that Lebesgue's dominated convergence Theorem concludes. $\square$

Lemma 17. *Let $\alpha \in (0,1)$, and $p, r \in [1, \infty]$.*

1. For all $u \in \widetilde{L}^r(0,T; \dot{B}_{p,\infty}^\alpha(\mathbb{R}^3))$,

$$\|\nabla u_\varepsilon\|_{L^r(0,T;L^p(\mathbb{R}^3))} = O(\varepsilon^{\alpha-1}) \quad and \quad \|\nabla^2 u_\varepsilon\|_{L^r(0,T;L^p(\mathbb{R}^3))} = O(\varepsilon^{\alpha-2}). \tag{136}$$

2. For all $u \in \widetilde{L}^r(0,T; \dot{B}_{p,c_0}^\alpha(\mathbb{R}^3))$,

$$\|\nabla u_\varepsilon\|_{L^r(0,T;L^p(\mathbb{R}^3))} = o(\varepsilon^{\alpha-1}) \quad and \quad \|\nabla^2 u_\varepsilon\|_{L^r(0,T;L^p(\mathbb{R}^3))} = o(\varepsilon^{\alpha-2}). \tag{137}$$

3. If $r, p \in [1, \infty)$, then for all $u \in L^r((0,T); L^p(\mathbb{R}^3))$,

$$\|\nabla u_\varepsilon\|_{L^r(0,T;L^p(\mathbb{R}^3))} = o(1/\varepsilon) \quad and \quad \|\nabla^2 u_\varepsilon\|_{L^r(0,T;L^p(\mathbb{R}^3))} = o(1/\varepsilon^2). \tag{138}$$

Proof. Let us only prove the part of (136) regarding ∇u_ε, the other cases being similar. We start from

$$\nabla u_\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \tilde{\psi}^\varepsilon(y)u(t,x-y)dy,$$

where

$$\tilde{\psi}^\varepsilon(y) := \varepsilon^{-3}\tilde{\psi}(\varepsilon^{-1}x), \text{ whith } \tilde{\psi} := \nabla\psi.$$

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Since the mean value of $\tilde{\psi}$ over $\mathbb{R}^3$ vanishes, we get

$$
\begin{aligned}
\nabla u_\varepsilon(t,x) &= \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\tilde{\psi}^\varepsilon(y)(u(t,x-y)-u(t,x))dy, \\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^3}|y|^\alpha\tilde{\psi}^\varepsilon(y)\frac{u(t,x-y)-u(t,x)}{|y|^\alpha}dy,
\end{aligned}
$$

and we conclude as in the proof of (132) above. $\qquad\square$

Let us now prove Lemma 15.

Proof of Lemma 15. In order to prove the part of the claim concerning the vector product it suffices to observe that $\mathcal{B}^\varepsilon[\phi^1,\phi^2]$ may be written

$$
\mathcal{B}^\varepsilon[\phi^1,\phi^2] = r^\varepsilon[\phi^1,\phi^2] - (\phi^1-\phi^1_\varepsilon)\times(\phi^2-\phi^2_\varepsilon),
$$

where

$$
r^\varepsilon[\phi^1,\phi^2](x) := \int_{\mathbb{R}^3}\psi^\varepsilon(y)\delta_y\phi^1(x)\times\delta_y\phi^2(x)\,dy.
$$

Now, if $\chi\in C_c^\infty((0,T)\times\mathbb{R}^3)$, there exists some nonnegative $\theta\in C_c^\infty((0,T)\times\mathbb{R}^3)$ taking the value 1 on $\mathrm{supp}(\chi)$, so that $\chi=\chi\theta^3$. Then, we have

$$
\int_0^T\int_{\mathbb{R}^3}\chi|r^\varepsilon[\phi^1,\phi^2]|\,|\nabla^i\phi^3_\varepsilon|\,dx\,dt =
$$

$$
\int_{(0,T)\times\mathbb{R}^3}\chi(t,x)\left|\int_{\mathbb{R}^3}\psi^\varepsilon(y)((\theta\delta_y\phi^1)(t,x))\times((\theta\delta_y\phi^2)(t,x))\,dy\right||(\theta\nabla^i\phi^3_\varepsilon)(t,x)|\,dt\,dx.
$$

But $\theta\delta_y\phi^1 = \delta_y(\theta\phi^1)-\phi^1\delta_y\theta$. Since $\mathrm{supp}(\psi^\varepsilon)$ is contained in some ball of size ε, and θ takes the value 1 on $\mathrm{supp}(\chi)$, for ε small enough, with y in $\mathrm{supp}(\psi^\varepsilon)$, $\delta_y\theta$ vanishes on $\mathrm{supp}(\chi)$:

$$
\text{for } \varepsilon \text{ small enough,} \quad \int_0^T\int_{\mathbb{R}^3}\chi|r^\varepsilon[\phi^1,\phi^2]|\,|\nabla^i\phi^3_\varepsilon|\,dx\,dt = \int_0^T\int_{\mathbb{R}^3}\chi|r^\varepsilon[\theta\phi^1,\theta\phi^2]|\,|\nabla^i(\theta\phi^3_\varepsilon)|\,dx\,dt.
$$

Then, use Hölder's inequality, estimating $r^\varepsilon[\theta\phi^1,\theta\phi^2]$ as in the proof of Lemma 16, and combine with Lemma 16 and Lemma 17, noticing that among the estimates, at least one of the O's is a o. $\qquad\square$

7 Vanishing of anomalous energy dissipation for the MLL equations: Proof of Part ii) of Theorem 7

In order to conclude the proof of Theorem 7, it is sufficient to prove that for all $\chi\in C_c^\infty((0,T)\times\mathbb{R}^3)$,

$$
\int_{(0,T)\times\mathbb{R}^3}\chi d_{\mathrm{MLL}}^{\mathfrak{a},\varepsilon}\,dx\,dt\to 0, \quad \text{as } \varepsilon\to 0. \tag{139}
$$

For the sequel, we fix such a function χ. We expand the local anomalous energy dissipation $d_{\mathrm{MLL}}^{\mathfrak{a},\varepsilon}$, defined in (58), into

$$
\begin{aligned}
d_{\mathrm{MLL}}^{\mathfrak{a},\varepsilon} &= -\mathcal{B}^\varepsilon[m,\partial_t m-2H]\cdot(\partial_t m_\varepsilon-2H_\varepsilon)+2\mathcal{B}^\varepsilon[m,\Delta m]\cdot(\partial_t m_\varepsilon-2H_\varepsilon) \\
&\quad +2\mathcal{B}^\varepsilon[m,\partial_t m-2H]\cdot\Delta m_\varepsilon-4\mathcal{B}^\varepsilon[m,\Delta m]\cdot\Delta m_\varepsilon,
\end{aligned}
$$

which we denote

$$
d_{\mathrm{MLL}}^{\mathfrak{a},\varepsilon} = F_1^\varepsilon[m,H]+F_2^\varepsilon[m,H]+F_3^\varepsilon[m,H]+F_4^\varepsilon[m].
$$

Now it suffices to proceed as follows.

Remark 3. *When $m \in \widetilde{L}_t^3 \dot{B}_{p,\infty}^\alpha \cap L_{t,x}^\infty$ with $\alpha \in (3/2, 2)$ and $p = 9/(3\alpha - 1)$, by interpolation m belongs to $\widetilde{L}_t^4 \dot{B}_{q,\infty}^\beta$ with $\beta := 3\alpha/4 \in (9/8, 3/2)$ and $q = 12/(4\beta - 1)$. This follows from the injection $L^\infty((0,T) \times \mathbb{R}^3) \hookrightarrow \widetilde{L}^3(0, T; B_{\infty,\infty}^0(\mathbb{R}^3))$ and Theorem 6.4.5 in [6].*

First term. Since $\partial_t m - 2H \in L^2((0,T) \times \mathbb{R}^3)$ and $m \in L^\infty((0,T) \times \mathbb{R}^3)$, we deduce from Lemma 15 (with $i = 0$, $\phi^1 = m$, $\phi^2 = \phi^3 = \partial_t m - 2H$) that

$$\int_{(0,T) \times \mathbb{R}^3} F_1^\varepsilon[m, H]\chi \, dx \, dt \to 0, \text{ as } \varepsilon \to 0. \tag{140}$$

Second term. For the second term we use first an integration by parts to get

$$\int_{(0,T) \times \mathbb{R}^3} \chi F_2^\varepsilon[m, H] \, dx \, dt = -2 \sum_k \int_{(0,T) \times \mathbb{R}^3} \chi \mathcal{B}^\varepsilon[m, \partial_k m] \cdot \partial_k(\partial_t m_\varepsilon - 2H_\varepsilon) \, dx \, dt$$

$$-2 \sum_k \int_{(0,T) \times \mathbb{R}^3} (\partial_k \chi) \mathcal{B}^\varepsilon[m, \partial_k m] \cdot (\partial_t m_\varepsilon - 2H_\varepsilon) \, dx \, dt$$

$$:= I_1^\varepsilon + I_2^\varepsilon.$$

Then, one uses Lemma 15 with $i = 1$,

$$\phi^3 = \partial_t m - 2H \in L^2((0,T) \times \mathbb{R}^3), \quad \phi^2 = \nabla m \in \widetilde{L}^4(0,T; \dot{B}_{q,\infty}^{\beta-1}(\mathbb{R}^3))_{\mathrm{loc}},$$

$$\text{and } \phi^1 = m \in \widetilde{L}^4(0,T; \dot{B}_{q,\infty}^\beta(\mathbb{R}^3))_{\mathrm{loc}} \hookrightarrow \widetilde{L}^4(0,T; \dot{B}_{\tilde{q},\infty}^{\tilde{\beta}}(\mathbb{R}^3))_{\mathrm{loc}}$$

for all $\tilde{q} \geqslant q$, with $\tilde{\beta} = \beta - 3\left(\frac{1}{q} - \frac{1}{\tilde{q}}\right)$. Under this last condition, $q = \frac{12}{4\beta - 1}$ is equivalent to $\tilde{q} = \frac{12}{4\tilde{\beta} - 1}$. It is then required that $\frac{1}{\tilde{q}} + \frac{1}{q} = \frac{1}{2}$, which leads, because of the relations $q = \frac{12}{4\beta - 1}$, $\tilde{q} = \frac{12}{4\tilde{\beta} - 1}$, to the constraint $\tilde{\beta} = 2 - \beta$. Note that, when β belongs to $(1, 2)$, $\tilde{\beta}$ and $\beta - 1$ belong to $(0, 1)$; furthermore, in this case, we have $q < 4 < \tilde{q}$. Now, the remaining requirement from Lemma 15 is $\tilde{\beta} + (\beta - 1) \geqslant 1$, which is fulfilled (since $\tilde{\beta} + (\beta - 1) = 1$). Hence, we get that $I_1^\varepsilon \to 0$ as $\varepsilon \to 0$.

Similarly, one uses Lemma 15 with the same functions ϕ^1, ϕ^2, ϕ^3, but this time with $i = 0$ and $\partial_k \chi$ instead of χ, to get that $I_2^\varepsilon \to 0$ as $\varepsilon \to 0$. Thus

$$\int_{(0,T) \times \mathbb{R}^3} \chi F_2^\varepsilon[m, H] \, dx \, dt \to 0, \text{ as } \varepsilon \to 0. \tag{141}$$

Third term. Now,

$$\int_{(0,T) \times \mathbb{R}^3} \chi F_3^\varepsilon[m, H] \, dx \, dt = 2 \sum_k \int_{(0,T) \times \mathbb{R}^3} \chi \mathcal{B}^\varepsilon[m, \partial_t m_\varepsilon - 2H_\varepsilon] \cdot \partial_k(\partial_k m) \, dx \, dt,$$

so that we apply again Lemma 15 with $i = 1$, $\phi^3 = \nabla m \in \widetilde{L}^4(0,T; \dot{B}_{q,\infty}^{\beta-1}(\mathbb{R}^3))_{\mathrm{loc}}$, $\phi^2 = \partial_t m - 2H \in L^2((0,T) \times \mathbb{R}^3)$ and $\phi^1 = m \in \widetilde{L}^4(0,T; \dot{B}_{q,\infty}^\beta(\mathbb{R}^3))_{\mathrm{loc}} \hookrightarrow \widetilde{L}^4(0,T; \dot{B}_{\tilde{q},\infty}^{\tilde{\beta}}(\mathbb{R}^3))_{\mathrm{loc}}$, exactly as for the second term $F_\varepsilon^2[m, H]$, to obtain

$$\int_{(0,T) \times \mathbb{R}^3} \chi F_3^\varepsilon[m, H] \, dx \, dt \to 0, \text{ as } \varepsilon \to 0. \tag{142}$$

Fourth term. Finally we use again an integration by parts to get

$$\int_{(0,T)\times\mathbb{R}^3} \chi F_4^\varepsilon[m]\,dx\,dt \;=\; 4\sum_k \int_{(0,T)\times\mathbb{R}^3} \chi \mathcal{B}^\varepsilon[m,\partial_k m]\cdot \Delta\partial_k m_\varepsilon\,dx\,dt$$

$$+\,4\sum_k \int_{(0,T)\times\mathbb{R}^3} (\partial_k\chi)\mathcal{B}^\varepsilon[m,\partial_k m]\cdot \Delta m_\varepsilon\,dx\,dt.$$

Let us start with the first term of the right hand side. Here, we invoke Lemma 15 with $i = 2$, so that the previous $\widetilde{L}^4 \dot{B}^\beta_{q,\infty}$-regularity for m is useless. Instead, we take $\phi^1 = m \in \widetilde{L}^3 \dot{B}^\alpha_{p,c_0} \hookrightarrow \widetilde{L}^3 \dot{B}^{\tilde\alpha}_{\tilde p,c_0}$, $\phi^2 = \phi^3 = \nabla m \in \widetilde{L}^3 \dot{B}^{\alpha-1}_{p,c_0}$. We have the constraints

$$\tilde\alpha = \alpha - 3\left(\frac{1}{p}-\frac{1}{\tilde p}\right) < \alpha, \quad \frac{1}{p}+\frac{2}{\tilde p}=1 \quad\text{and}\quad \tilde\alpha + 2(\alpha-1) \geqslant 2.$$

Choosing $\tilde\alpha + 2(\alpha-1) = 2$ is equivalent to the relation $p = \frac{9}{3\alpha-1}$. Furthermore, imposing $\tilde\alpha = 4-2\alpha \in (0,1)$ is equivalent to $\alpha \in (3/2,2)$. This is enough to ensure

$$\int_{(0,T)\times\mathbb{R}^3} \chi F_4^\varepsilon[m]\,dx\,dt \to 0, \text{ as } \varepsilon \to 0. \tag{143}$$

Gathering (140)-(141)-(142)-(143) yields (139).

8 Vanishing of anomalous dissipations for the HMHD equations: Proof of Part ii) of Theorem 8 and Theorem 9

8.1 No anomalous magneto-helicity dissipation: Proof of Part ii) of Theorem 8

In order to prove Part ii) of Theorem 8, it is sufficient to prove that for all $\chi \in C_c^\infty((0,T)\times\mathbb{R}^3)$,

$$\int_{(0,T)\times\mathbb{R}^3} \chi d_m^{\mathfrak{a},\varepsilon}\,dx\,dt \to 0, \quad\text{as } \varepsilon \to 0. \tag{144}$$

For the sequel, we fix such a function χ. Let us recall the definition of $d_m^{\mathfrak{a},\varepsilon}$ given in (64):

$$d_m^{\mathfrak{a},\varepsilon} := 2A_\varepsilon \cdot \operatorname{curl} \mathcal{B}^\varepsilon[u,B] - 2A_\varepsilon \cdot \operatorname{curl}\operatorname{div} \mathcal{C}^\varepsilon[B,B] =: 2T_1^\varepsilon[u,A] - 2T_2^\varepsilon[A].$$

We use (109) and an integration by parts to obtain:

$$\int_{(0,T)\times\mathbb{R}^3} \chi T_1^\varepsilon[u,A]\,dx\,dt \;=\; \int_{(0,T)\times\mathbb{R}^3} \chi B_\varepsilon \cdot \mathcal{B}^\varepsilon[u,B]\,dx\,dt \tag{145}$$

$$-\int_{(0,T)\times\mathbb{R}^3} \big(\mathcal{B}^\varepsilon[u,B]\times A_\varepsilon\big)\cdot\nabla\chi\,dx\,dt.$$

On the other hand, we apply (109) to get:

$$\int_{(0,T)\times\mathbb{R}^3} \chi T_2^\varepsilon[A]\,dx\,dt \;=\; \int_{(0,T)\times\mathbb{R}^3} \chi \operatorname{curl} A_\varepsilon \cdot \operatorname{div} \mathcal{C}^\varepsilon[B,B]\,dx\,dt$$

$$+\int_{(0,T)\times\mathbb{R}^3} \chi \operatorname{div}\Big(\big(\operatorname{div}\mathcal{C}^\varepsilon[B,B]\big)\times A_\varepsilon\Big)\,dx\,dt.$$

An integration by parts yields

$$\int_{(0,T)\times\mathbb{R}^3} \chi T_2^\varepsilon[A]\,dx\,dt = -\int_{(0,T)\times\mathbb{R}^3} \chi(\nabla\operatorname{curl}A_\varepsilon) : \mathcal{C}^\varepsilon[B,B]\,dx\,dt$$
$$-\int_{(0,T)\times\mathbb{R}^3} (\mathcal{C}^\varepsilon[B,B]\operatorname{curl}A_\varepsilon)\cdot\nabla\chi\,dx\,dt$$
$$-\int_{(0,T)\times\mathbb{R}^3} \big((\operatorname{div}\mathcal{C}^\varepsilon[B,B])\times A_\varepsilon\big)\cdot\nabla\chi\,dx\,dt,$$

so that we obtain finally:

$$\int_{(0,T)\times\mathbb{R}^3} \chi T_2^\varepsilon[A]\,dx\,dt = -\int_{(0,T)\times\mathbb{R}^3} \chi(\nabla B_\varepsilon) : \mathcal{C}^\varepsilon[B,B]\,dx\,dt$$
$$-\int_{(0,T)\times\mathbb{R}^3} (\mathcal{C}^\varepsilon[B,B]B_\varepsilon)\cdot\nabla\chi\,dx\,dt$$
$$+\int_{(0,T)\times\mathbb{R}^3} \Big((\mathcal{C}^\varepsilon[B,B]\nabla)\times A_\varepsilon\Big)\cdot\nabla\chi\,dx\,dt$$
$$+\sum_j\int_{(0,T)\times\mathbb{R}^3} \Big(((\mathcal{C}^\varepsilon[B,B])_{ij})_i\times A_\varepsilon\Big)\cdot\partial_j\nabla\chi\,dx\,dt. \tag{146}$$

To get the vanishing of the T_1^ε term given by (145) as ε goes to zero, observe that the regularity given by Theorem 2 and interpolation theory suffices, applying Lemma 15 with $i=0$ and the ϕ^j's equal to u, B or A, all belonging to $L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))\hookrightarrow L^3(0,T;L^3(\mathbb{R}^3))$.

Concerning the last three terms produced by T_2^ε in (146), use again Lemma 15 with $i=0$, $\phi^1=\phi^2=B$, and ϕ^3 being B, ∇A or A, which all belong to $L^3(0,T;L^3(\mathbb{R}^3))$. The first term in (146) is the one for which the most regularity is needed: in Lemma 15, take $i=1$ and $\phi^1=\phi^2=\phi^3=B\in\widetilde{L}^3(0,T;\dot{B}_{3,c_0}^{1/3}(\mathbb{R}^3))_{\mathrm{loc}}$.

8.2 No anomalous energy dissipation: Proof of Part ii) of Theorem 9

In order to prove Part ii) of Theorem 9, we consider $\chi\in C_c^\infty((0,T)\times\mathbb{R}^3)$, and we prove that

$$\int_{(0,T)\times\mathbb{R}^3} \chi d_{\mathrm{HMHD}}^{\mathfrak{a},\varepsilon}\,dx\,dt \to 0, \quad \text{as } \varepsilon\to 0. \tag{147}$$

We recall that

$$d_{\mathrm{HMHD}}^{\mathfrak{a},\varepsilon} := -u_\varepsilon\cdot\operatorname{div}\Big(\mathcal{C}^\varepsilon[u,u]-\mathcal{C}^\varepsilon[B,B]\Big) - \frac{1}{2}u_\varepsilon\cdot\nabla\mathcal{A}^\varepsilon[B,B]$$
$$-B_\varepsilon\cdot\Big(\operatorname{curl}\mathcal{B}^\varepsilon[u,B]\Big) + B_\varepsilon\cdot\Big(\operatorname{curl}\operatorname{div}\mathcal{C}^\varepsilon[B,B]\Big)$$
$$=: J_1^\varepsilon[u,B] + J_2^\varepsilon[u,B] + J_3^\varepsilon[u,B] + J_4^\varepsilon[B].$$

Using (109) and that u_ε is divergence free, we have

$$\int_{(0,T)\times\mathbb{R}^3} \chi J_1^\varepsilon[u,B]\,dx\,dt = \int_{(0,T)\times\mathbb{R}^3} \chi\nabla u_\varepsilon\cdot\Big(\mathcal{C}^\varepsilon[u,u]-\mathcal{C}^\varepsilon[B,B]\Big)\,dx\,dt$$
$$+\int_{(0,T)\times\mathbb{R}^3} \Big(\mathcal{C}^\varepsilon[u,u]-\mathcal{C}^\varepsilon[B,B]\Big)(u_\varepsilon\cdot\nabla\chi)\,dx\,dt,$$

$$\int_{(0,T)\times\mathbb{R}^3} \chi J_2^\varepsilon[u,B]\,dx\,dt = \frac{1}{2}\int_{(0,T)\times\mathbb{R}^3} (u_\varepsilon\cdot\nabla\chi)\mathcal{A}^\varepsilon[B,B]\,dx\,dt,$$

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$$\int_{(0,T)\times\mathbb{R}^3} \chi J_3^\varepsilon[u,B]\,dx\,dt \;=\; -\int_{(0,T)\times\mathbb{R}^3} \chi(\operatorname{curl} B_\varepsilon)\cdot\mathcal{B}^\varepsilon[u,B]\,dx\,dt$$

$$-\int_{(0,T)\times\mathbb{R}^3}\left(B_\varepsilon\times\mathcal{B}^\varepsilon[u,B]\right)\cdot\nabla\chi\,dx\,dt,$$

and

$$\int_{(0,T)\times\mathbb{R}^3}\chi J_4^\varepsilon[B]\,dx\,dt \;=\; \int_{(0,T)\times\mathbb{R}^3}\chi\operatorname{div}\left((\operatorname{div}\mathcal{C}^\varepsilon[B,B])\times B_\varepsilon\right)dx\,dt$$

$$+\int_{(0,T)\times\mathbb{R}^3}\chi\left((\operatorname{div}\mathcal{C}^\varepsilon[B,B])\cdot\operatorname{curl} B_\varepsilon\right)dx\,dt,$$

by using again (109).

Then, integrating by parts:

$$\int_{(0,T)\times\mathbb{R}^3}\chi J_4^\varepsilon[B]\,dx\,dt \;=\; -\int_{(0,T)\times\mathbb{R}^3}\left((\operatorname{div}\mathcal{C}^\varepsilon[B,B])\times B_\varepsilon\right)\cdot\nabla\chi\,dx\,dt$$

$$-\int_{(0,T)\times\mathbb{R}^3}\chi\nabla(\operatorname{curl} B_\varepsilon):\mathcal{C}^\varepsilon[B,B]\,dx\,dt$$

$$-\int_{(0,T)\times\mathbb{R}^3}\left(\mathcal{C}^\varepsilon[B,B]\operatorname{curl} B_\varepsilon\right)\cdot\nabla\chi,$$

so that

$$\;=\; \int_{(0,T)\times\mathbb{R}^3}\left((\mathcal{C}^\varepsilon[B,B]\nabla)\times B_\varepsilon\right)\cdot\nabla\chi\,dx\,dt$$

$$+\sum_j\int_{(0,T)\times\mathbb{R}^3}\left(((\mathcal{C}^\varepsilon[B,B])_{ij})_i\times B_\varepsilon\right)\cdot\partial_j\nabla\chi\,dx\,dt$$

$$-\int_{(0,T)\times\mathbb{R}^3}\chi\nabla(\operatorname{curl} B_\varepsilon):\mathcal{C}^\varepsilon[B,B]\,dx\,dt \tag{148}$$

$$-\int_{(0,T)\times\mathbb{R}^3}\left(\mathcal{C}^\varepsilon[B,B]\operatorname{curl} B_\varepsilon\right)\cdot\nabla\chi\,dx\,dt.$$

Finally, we use repetitively Lemma 15 with the ϕ^j's equal to u or B, and $i=0$ ($u,B\in L^3(0,T;L^3(\mathbb{R}^3))$), $i=1$ ($u,B\in\widetilde{L}^3(0,T;\dot{B}_{3,c_0}^{1/3}(\mathbb{R}^3))_{\mathrm{loc}}$), or $i=2$ ($u,B\in\widetilde{L}^3(0,T;\dot{B}_{3,c_0}^{2/3}(\mathbb{R}^3))_{\mathrm{loc}}$): the worse term to cope with, in term of needed regularity, is $\chi\nabla(\operatorname{curl} B_\varepsilon):\mathcal{C}^\varepsilon[B,B]$, from (148). It then follows that, as $\varepsilon\to 0$,

$$\text{for } i=1,2,3, \quad \int_{(0,T)\times\mathbb{R}^3}\chi J_i^\varepsilon[u,B]\,dx\,dt\to 0, \quad\text{and}\quad \int_{(0,T)\times\mathbb{R}^3}\chi J_4^\varepsilon[B]\,dx\,dt\to 0,$$

which provides (147).

9 Vanishing of anomalous crossed fluid-magneto-helicity dissipation for the MHD equations: Proof of Part iii) of Theorem 11

We consider the inner product of equation (125) with B_ε and the inner product of equation (126) with u_ε and we sum the resulting identities, taking into account that

$$B_\varepsilon \cdot \operatorname{div}(u_\varepsilon \otimes u_\varepsilon - B_\varepsilon \otimes B_\varepsilon) - u_\varepsilon \cdot \operatorname{curl}(u_\varepsilon \times B_\varepsilon) = \operatorname{div}\left((u_\varepsilon \cdot B_\varepsilon)u_\varepsilon - \frac{1}{2}(|u_\varepsilon|^2 + |B_\varepsilon|^2)B_\varepsilon\right),$$

$$B_\varepsilon \cdot \nabla(p_\varepsilon + \frac{1}{2}|B_\varepsilon|^2) = \operatorname{div}\left((p_\varepsilon + \frac{1}{2}|B_\varepsilon|^2)B_\varepsilon\right),$$

$$-B_\varepsilon \cdot \Delta u_\varepsilon - u_\varepsilon \cdot \Delta B_\varepsilon = 2(\operatorname{curl} u_\varepsilon) \cdot (\operatorname{curl} B_\varepsilon) + \operatorname{div}\left((\operatorname{curl} u_\varepsilon) \times B_\varepsilon\right) + \operatorname{div}\left((\operatorname{curl} B_\varepsilon) \times u_\varepsilon\right).$$

We obtain that

$$\partial_t h^\varepsilon_{fm} + d^\varepsilon_{fm} + \operatorname{div} f^\varepsilon_{fm} = d^{\mathfrak{a},\varepsilon}_{fm},$$

where

$$h^\varepsilon_{fm} := u_\varepsilon \cdot B_\varepsilon, \quad d^\varepsilon_{fm} := 2\omega_\varepsilon \cdot \operatorname{curl} B_\varepsilon,$$

$$f^\varepsilon_{fm} := (u_\varepsilon \cdot B_\varepsilon)u_\varepsilon + (p_\varepsilon - \frac{1}{2}|u_\varepsilon|^2)B_\varepsilon + (\operatorname{curl} u_\varepsilon) \times B_\varepsilon + (\operatorname{curl} B_\varepsilon) \times u_\varepsilon,$$

$$d^{\mathfrak{a},\varepsilon}_{fm} := -B_\varepsilon \cdot \operatorname{div}\left(\mathcal{C}^\varepsilon[u,u] - \mathcal{C}^\varepsilon[B,B]\right) - \frac{1}{2}B_\varepsilon \cdot \nabla \mathcal{A}^\varepsilon[B,B] + u^\varepsilon \cdot \operatorname{curl} \mathcal{B}^\varepsilon[u,B].$$

One easily sees that h^ε_{fm}, d^ε_{fm}, and f^ε_{fm} converge respectively in the sense of distributions to h_{fm}, d_{fm} and f_{fm}.

Finally, we use repetitively Lemma 15 to conclude that the anomalous dissipation $d^{\mathfrak{a},\varepsilon}_{fm}$ vanishes when $\varepsilon \to 0$.

10 Suitable solutions: Proof of Theorem 12 and of Theorem 13

10.1 Proof of Theorem 12

We recall from [2, 10] that the weak maximum principle yields that for any $\varepsilon \in (0,1)$, m^ε is bounded by 1 almost everywhere in space and time. Then, using equation (69) and the estimate (73) we deduce that m^ε belongs to the space $L^2((0,T); H^2(\mathbb{R}^3; \mathbb{R}^3))$ (observe however that this provides an estimate of the norm of m^ε in this space which is not uniform in ε).

Now, observe that, formally, multiplying (69) by $\partial_t m^\varepsilon$, (70) by H^ε, (71) by E^ε, and summing the resulting identities lead to

$$\partial_t\left(e^\varepsilon_{\mathrm{MLL}} + e^\varepsilon_{\mathrm{GL}}\right) + d^\varepsilon_{\mathrm{MLL}} + \operatorname{div} f^\varepsilon_{\mathrm{MLL}} = -2(m^\varepsilon \cdot H^\varepsilon)(m^\varepsilon \cdot \partial_t m^\varepsilon), \tag{149}$$

where $(e^\varepsilon_{\mathrm{MLL}}, d^\varepsilon_{\mathrm{MLL}}, f^\varepsilon_{\mathrm{MLL}})$ is given by (120) and $e^\varepsilon_{\mathrm{GL}}$ by (74).

In fact, Identity (149) can be easily justified. On one hand the smoothness of $(m^\varepsilon, H^\varepsilon, E^\varepsilon)$ is sufficient to manipulate all the terms coming from the multiplication of (69) by $\partial_t m^\varepsilon$. On the other hand, one can mollify the linear equations (70) and (71) and then multiply them respectively by H^ε and E^ε. It then remains to sum the resulting identities and to pass to the limit with respect to the regularization parameter. Thus, the local energy identity (149) holds true without any anomalous dissipation.

Now, using the uniform bounds of $(m^\varepsilon, H^\varepsilon, E^\varepsilon)$ provided by (73), we have, up to a subsequence: $f^\varepsilon_{\mathrm{MLL}}$ converges in the sense of distributions to f_{MLL}; $d^\varepsilon_{\mathrm{MLL}}$ and $e^\varepsilon_{\mathrm{MLL}} + e^\varepsilon_{\mathrm{GL}}$ converge in the sense of distributions

respectively to some distributions $\tilde{d}^{\mathfrak{a},1}_{\text{MLL}}$ and $\tilde{e}^{\mathfrak{a}}_{\text{MLL}}$ such that, by lower weak semicontinuity, $\tilde{d}^{\mathfrak{a},1}_{\text{MLL}} - |\partial_t m|^2$ and $\tilde{e}^{\mathfrak{a}}_{\text{MLL}} - e_{\text{MLL}}$ are non negative. Finally the term $(m^\varepsilon \cdot H^\varepsilon)(m^\varepsilon \cdot \partial_t m^\varepsilon)$ converges in the sense of distributions to 0.

Therefore it follows from (57) that the anomalous energy dissipation $d^{\mathfrak{a}}_{\text{MLL}}$ is given by $d^{\mathfrak{a}}_{\text{MLL}} = -d^{\mathfrak{a},1}_{\text{MLL}} - \partial_t e^{\mathfrak{a}}_{\text{MLL}}$, where $d^{\mathfrak{a},1}_{\text{MLL}} := \tilde{d}^{\mathfrak{a},1}_{\text{MLL}} - d_{\text{MLL}}$ and $e^{\mathfrak{a}}_{\text{MLL}} := \tilde{e}^{\mathfrak{a}}_{\text{MLL}} - e_{\text{MLL}}$ are non negative.

10.2 Proof of Theorem 13

Multiplying (75) by u^ε, (77) by B^ε, and summing the resulting identities lead to

$$\partial_t e^\varepsilon_{\text{HMHD}} + d^\varepsilon_{\text{HMHD}} + \operatorname{div} \tilde{f}^\varepsilon_{\text{HMHD}} = 0, \tag{150}$$

where $e^\varepsilon_{\text{HMHD}}$ and $d^\varepsilon_{\text{HMHD}}$ are given by the formula in (129) and

$$\tilde{f}^\varepsilon_{\text{HMHD}} := \frac{1}{2}|u^\varepsilon|^2 (u^\varepsilon)_\varepsilon + p^\varepsilon u^\varepsilon + B^\varepsilon \times (u^\varepsilon \times (B^\varepsilon)_\varepsilon) + (\operatorname{curl} u^\varepsilon) \times (u^\varepsilon)_\varepsilon$$
$$+ (\operatorname{curl} B^\varepsilon) \times (B^\varepsilon)_\varepsilon + \big((\operatorname{curl} B^\varepsilon) \times (B^\varepsilon)_\varepsilon\big) \times B^\varepsilon.$$

We do not detail the computations here since it is sufficient to adapt what we have already done in Section 5.2.3 in order to obtain (128) with here an extra bookkeeping of the mollifications.

Now, using that, up to a subsequence, u^ε and B^ε converges in $L^3((0,T) \times \mathbb{R}^3)_{\text{loc}}$ respectively to u and B, and therefore $(u^\varepsilon)_\varepsilon$ and $(B^\varepsilon)_\varepsilon$ converges as well in $L^3((0,T) \times \mathbb{R}^3)_{\text{loc}}$ respectively to u and B, and that $\operatorname{curl} u^\varepsilon$ and $\operatorname{curl} B^\varepsilon$ weakly converge to $\operatorname{curl} u$ and $\operatorname{curl} B$ in $L^2((0,T) \times \mathbb{R}^3)$, we get that

$$\frac{1}{2}\partial_t \Big(|u^\varepsilon|^2 + |B^\varepsilon|^2\Big) + \operatorname{div} \Big((\frac{1}{2}|u^\varepsilon|^2 (u^\varepsilon)_\varepsilon + B^\varepsilon \times (u^\varepsilon \times (B^\varepsilon)_\varepsilon) + (\operatorname{curl} u^\varepsilon) \times (u^\varepsilon)_\varepsilon + (\operatorname{curl} B^\varepsilon) \times (B^\varepsilon)_\varepsilon\Big)$$

converges in the sense of distributions to

$$\frac{1}{2}\partial_t \Big(|u|^2 + |B|^2\Big) + \operatorname{div} \Big(\frac{1}{2}|u|^2 u + B \times (u \times B) + (\operatorname{curl} u) \times u + (\operatorname{curl} B) \times B\Big).$$

Moreover, using once again (130), we get that p_ε converges in $L^{\frac{3}{2}}((0,T) \times \mathbb{R}^3)_{\text{loc}}$ to p. Thus $\operatorname{div}(p^\varepsilon u^\varepsilon)$ converges in the sense of distributions to $\operatorname{div}(pu)$.

Finally, using that, up to a subsequence, B^ε converges to B in $L^4((0,T) \times \mathbb{R}^3)_{\text{loc}}$ we obtain that

$$\operatorname{div} \big((\operatorname{curl} B^\varepsilon) \times (B^\varepsilon)_\varepsilon) \times B^\varepsilon\big) \tag{151}$$

converges in the sense of distributions to $\operatorname{div} \big((\operatorname{curl} B) \times B) \times B\big)$.

We therefore get that the anomalous energy dissipation $d^{\mathfrak{a}}_{\text{HMHD}}$ is given by the discrepancy between the limit of $|\operatorname{curl} u^\varepsilon|^2 + |\operatorname{curl} B^\varepsilon|^2$ and $|\operatorname{curl} u|^2 + |\operatorname{curl} B|^2$ which is non positive by lower weak semicontinuity.

Since the term (151) vanishes when the Hall effect is omitted one sees that the assumption that, up to a subsequence, B^ε converges to B in $L^4((0,T) \times \mathbb{R}^3)_{\text{loc}}$, is not needed in order to prove the result for the MHD equations.

Appendix: proof of Lemma 14

Here, we prove that for all $p, r \in [1, \infty]$, $\alpha \in (0,1)$ and $\tilde{\alpha} \in (0, \alpha)$, with $\tilde{p}$ defined by $\tilde{\alpha} - 3/\tilde{p} = \alpha - 3/p$, the space $\widetilde{L}^r(0, T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))$ is continuously embedded in $\widetilde{L}^r(0, T; \dot{B}^{\tilde{\alpha}}_{\tilde{p},\infty}(\mathbb{R}^3))$.

The proof of the other case, when $\alpha \in (1,2)$ and $\tilde{\alpha} \in (0,1) \cup (1,\alpha)$, follows, since, when $\alpha \in (1,2)$,

$$\|u\|_{\widetilde{L}^r(0,T;\dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))} = \|u\|_{\widetilde{L}^r(0,T;\dot{B}^{\alpha-1}_{p,\infty}(\mathbb{R}^3))} + \sum_i \|\partial_i u\|_{\widetilde{L}^r(0,T;\dot{B}^{\alpha-1}_{p,\infty}(\mathbb{R}^3))}.$$

As usual, $A \lesssim B$ denotes the inequality $A \leqslant CB$ for some universal constant C. We recall the existence of a smooth dyadic partition of unity: there exists a smooth radial function φ, supported in the annulus $C(3/4, 8/3) := \{3/4 < |\xi| < 8/3\}$, with values in the interval $[0, 1]$, such that

$$\forall \xi \in \mathbb{R}^3 \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1; \qquad |j - j'| \geqslant 2 \Rightarrow \text{supp } \varphi(2^{-j}\cdot) \cap \text{ supp } \varphi(2^{-j'}\cdot) = \emptyset.$$

The so-called dyadic blocks $\dot{\Delta}_j$ correspond to the Fourier multipliers $\dot{\Delta}_j := \varphi(2^{-j}D)$, that is

$$\dot{\Delta}_j u(x) := 2^{3j} \int_{\mathbb{R}^3} h(2^j y) u(x - y) dy \quad \text{for } j \in \mathbb{Z}, \quad \text{where } h := \mathcal{F}^{-1}\varphi.$$

Then, for all $u \in \mathcal{S}'_h$ (from Definition 6), homogeneous Littlewood-Paley decomposition holds: $u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u$.

Let $u \in \widetilde{L}^r(0, T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))$ and $y \in \mathbb{R}^3 \setminus \{0\}$. Denoting $\delta_y u(t, x) = u(t, x - y) - u(t, x)$, we write

$$\delta_y u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j \delta_y u.$$

Thus, we estimate

$$\|\|\delta_y u\|_{L^{\tilde{p}}(\mathbb{R}^3)}\|_{L^r(0,T)} \leqslant \sum_{j \in \mathbb{Z}} \|\|\delta_y \dot{\Delta}_j u\|_{L^{\tilde{p}}(\mathbb{R}^3)}\|_{L^r(0,T)},$$

which splits up, for any $j_y \in \mathbb{Z}$, as the sum $I + II$, where

$$I = \sum_{j \leqslant j_y} \|\|\delta_y \dot{\Delta}_j u\|_{L^{\tilde{p}}(\mathbb{R}^3)}\|_{L^r(0,T)} \quad \text{and} \quad II = \sum_{j > j_y} \|\|\delta_y \dot{\Delta}_j u\|_{L^{\tilde{p}}(\mathbb{R}^3)}\|_{L^r(0,T)}.$$

We recall that Bernstein's lemma implies

$$\|\dot{\Delta}_j u\|_{L^{\tilde{p}}(\mathbb{R}^3)} \lesssim 2^{3j\left(\frac{1}{p} - \frac{1}{\tilde{p}}\right)} \|\dot{\Delta}_j u\|_{L^p(\mathbb{R}^3)} = 2^{3j(\alpha - \tilde{\alpha})} \|\dot{\Delta}_j u\|_{L^p(\mathbb{R}^3)}. \tag{152}$$

We also notice that

$$\|\delta_y \dot{\Delta}_j u\|_{L^{\tilde{p}}(\mathbb{R}^3)} \leqslant 2\|\dot{\Delta}_j u\|_{L^{\tilde{p}}(\mathbb{R}^3)}, \tag{153}$$

as well as

$$\|\delta_y \dot{\Delta}_j u\|_{L^{\tilde{p}}(\mathbb{R}^3)} \lesssim 2^j |y| \sum_{|j-j'|\leqslant 1} \|\Delta_{j'} u\|_{L^{\tilde{p}}(\mathbb{R}^3)} \tag{154}$$

(see [4], page 75), and

$$2^{j\alpha} \|\|\delta_y \dot{\Delta}_j u\|_{L^{\tilde{p}}(\mathbb{R}^3)}\|_{L^r(0,T)} \lesssim \|u\|_{\widetilde{L}^r(0,T;\dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))} \tag{155}$$

(from [4], page 76).

Then, for any term in the sum I, we have

$$
\begin{aligned}
\|\|\delta_y \dot{\Delta}_j u\|_{L^{\tilde{p}}(\mathbb{R}^3)}\|_{L^r(0,T)} &\lesssim 2^j |y| \sum_{|j-j'|\leqslant 1} \|\|\Delta_{j'} u\|_{L^{\tilde{p}}(\mathbb{R}^3)}\|_{L^r(0,T)} \quad \text{by (154)}, \\
&\lesssim 2^j |y| 2^{3j(\alpha - \tilde{\alpha})} \sum_{|j-j'|\leqslant 1} \|\|\Delta_{j'} u\|_{L^p(\mathbb{R}^3)}\|_{L^r(0,T)} \quad \text{by (152)}, \\
&\lesssim |y| 2^{j(1-\tilde{\alpha})} \|u\|_{\widetilde{L}^r(0,T;\dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))} \quad \text{by (155)}.
\end{aligned}
\tag{156}
$$

Now, for the terms in the sum II, inequalities (153), (152) and (155) lead to

$$\|\|\delta_y \dot{\Delta}_j u\|_{L^{\tilde{p}}(\mathbb{R}^3)}\|_{L^r(0,T)} \lesssim 2^{-j\tilde{\alpha}}\|u\|_{\widetilde{L}^r(0,T;\dot{B}^{\alpha}_{p,\infty}(\mathbb{R}^3))}. \tag{157}$$

Finally, choosing $j_y \in \mathbb{Z}$ such that $\dfrac{1}{|y|} \leqslant 2^{j_y} \dfrac{2}{|y|}$, we get $|y| \sum\limits_{j \leqslant j_y} 2^{j(1-\tilde{\alpha})} + \sum\limits_{j > j_y} 2^{-j\tilde{\alpha}} \lesssim |y|^{\tilde{\alpha}}$, and hence the desired inequality,

$$\|u\|_{\widetilde{L}^r(0,T;\dot{B}^{\tilde{\alpha}}_{\tilde{p},\infty}(\mathbb{R}^3))} \lesssim \|u\|_{\widetilde{L}^r(0,T;\dot{B}^{\alpha}_{p,\infty}(\mathbb{R}^3))}.$$

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